[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: OK, so let me just recall first that we were talking about a very important concept of a sequence, concept of a sequence. So a sequence, the formal definition is that you have a map from the natural number into R. So then the image of a particular natural number n, it's usually denoted like an, like this. It could be some other instead of a, it could be bn.

Potentially, it could be another subscript. But this is the most common thing to denote it like this. But it really is a function. So that's a sequence.

So a subsequence is that you have a sequence and then you have another map. But this map is not just any map from the integers to the integers. It has to be strictly increasing. And why does it have to be strictly increasing? Because we want, for the subsequence, if you think about the numbers that come out when you plug in a, when you're looking at the an, then you want some of those numbers to be in the subsequence.

But you want the ordering to be the same. So that's why this map has to be strictly increasing. You can't pick the several elements many times if it wasn't picked many times in the original sequence. So a subsequence is where you have another map from the natural number to the natural number that is strictly increasing.

And then you're looking at the composition of these two maps. So you're looking at this in here. And you call this here. And then that's usually denoted something like this-- ank. An is the original sequence, k means that it's really that you take the image of k under this strictly increasing map.

And then you're looking at the element in the sequence. So, nk, really, you should think about it as g of k. So that's what a subsequence is. And again, it's important that it's strictly increasing because you want the elements to be in the same order than it was in the original sequence. And you don't even want things to be picked more times than it was in the original sequence.

So this whole concept of a sequence and subsequence is really important. Then, the next really important concept is of convergence. So if you have a sequence an, then you say that it's converging to a. So a is another real number. It's not infinity. It's another real number. And so an converges to a if for all epsilon greater than 0, there exist N, an integer N, denoted by capital N, such that if little n is bigger or equal to this capital N, then an minus a have to be less than epsilon.

So if you are sufficiently far out in the sequence, then everything in that sequence is bunched epsilon close to n. That's what it's saying. And so if you have a sequence, a sequence that is not convergent, it is said to be divergent.

And the next thing, I'm just briefly reviewing this from last time-- the next thing for sequences are the algebraic laws or whatever you want to call them. And so that is that if you take a sequence, if you have a sequence an and an converts to a, bn converts to b, then if you form the sum of these two, an plus bn, so this is a new sequence where the n-th element is the sum of the n-th element of the other two sequences. Then this sequence here cn also converges.

And the limit here is the sum of the limits. And likewise, if you define a sequence-- so this was 1-- 2 is that if you take a constant. So this is a constant. And you define cn to be c times an, then this cn also converge. And the limit here is just c times a. That's the second one. I might have done it in the other order. But there's no particular order to have it in.

The third one is that if you are looking at the products. So if you define a sequence that is given by the product of the two, then cn converge also. And the limit is the product of the limits. And then the fourth rule is that if an is not equal to 0 for all n and the limit here is also not equal to 0, then if you're looking at 1 over an, then those converge to 1 over a.

So those are the four rules. Let me just write it as 4. And again, it's not like there's any particular convention with the ordering. And I think I might have exchanged the order of 1 and 2 last time. Now, let's see an example of how that works. Let's define a sequence n squared plus 1, n squared plus n plus 1.

Let's look at this sequence here for each n that gives us, actually, a rational number, but it's a real number. Now in order to understand this, I can rewrite this one here, as I can factor out n squared here. So I can write it like this. Like that. And then, of course, I can get rid of the n squared in both numerator and denominator.

So you may as well think about the sequence as this thing here. So now, I can just first look at the sequence which is 1 over n. If I'm looking at this sequence here, 1 over n, this here converges to 0. This essentially just follows from the Archimedean property.

But we've already seen this, that 1 over n converges to 0. And the same holds for 1 over n squared. This also converges to 0. And so if you're looking at the sequence, you could even look at the sequence, let's call this xn. This here is yn. I'm going to run out of character very quickly, zn. This is just going to be the sequence, which is constant 1.

Well, if it's constant 1, then obviously it's converging to 1. So now I can think about the sequence being to be equal to z plus y, zn plus yn. This here is a sequence. Here is what yn is. Here is what is zn is. So if you add these two, this is converging to 1. This is converging to 0.

By the algebraic property of limits, this converge also to the sum of the limit. So this converges to 1. Now, I can also define a sequence where I'm just adding zn and then I'm adding xn and then I'm adding yn. This here converges to 1, this here to 0, this here to 0. So the sum here also converge to the sum of the limit. But the sum of the limit is 1.

Put it up here. And then I'll just go over here. So now you have that cn converge to 1. And let's call this c. And so, in particular, all of these numbers here, these things here are all strictly positive. So each of the cn's are strictly positive. So they're definitely not 0. So the cn, so not only do they converge to 1, but the cn's are not equal to 0.

And the limit c, well, that's 1. So that's, of course, not equal to 0 either. So you can apply the fourth rule here. The fourth rule for the sequence, for the sequence cn, so applying the fourth rule to the sequence cn, and then you have that 1 over cn converge to 1 over c, but c was 1, so this is 1.

Now you can look at bn times 1 over cn. But bn times 1 over cn-- remember what bn was. Here's bn. And bn is really the numerator up here. It's the numerator up here when you have divided by n squared on both in the top and the numerator. So bn is the numerator and cn is the denominator. So bn divided by cn, this is an.

And now this here converge. That's what we saw over there. Using the algebraic rule, it converges to 1. This here also converge to 1 by the algebraic rules. Number 3 of the algebraic rules is that for the product it will also converge. And it converges to the product of the limits. So that's 1.

Obviously, this is like when you're looking at a sequence, it's often expressed in some form with some functions like this. And you do this reduction. And very quickly, this would be almost automatic. This is too easy almost, this one. This is the first time you see it, but you know in a couple of weeks this will feel too easy. Then there will be something more complicated.

So that's the algebraic rules for sequences. The next thing I want to talk about is, I want to try to make more precise this about square. We made it pretty precise, but I want to think about-- I don't have a very good eraser, unfortunately. Let me just see. This is a better one. This one here will make it very dirty.

So the next thing, we did define square root of 2. And we talked about that quite extensively, how square root of 2, how it really was defined. And it was a real number. And the real numbers followed from that, the field, the reals is complete.

But we also, in the back of our mind, we had that we already sort of knew what square root of 2 were because when you think about square root of 2, at least I think about square root of 2, I think about it as 1.4. Just because you can't really express it as a fraction. So you think about it as 1.414, et cetera. And then I don't remember the second, third, and fourth, whatever. But you may remember just a few of the digits.

So that's sort of our intuition with 1.4-- with square root 2. But how do we make this more precise? And so to make this more precise, we want to look at particular sequences, what's called monotone sequences. So what is a monotone sequence? So there are two types of monotone sequences. There's one which is monotone increasing sequence.

And then there's, similarly, monotone decreasing sequence. So a monotone increasing sequence, this just means that for all n, you have that the next element is larger or equal to the previous element. That's a monotone increasing sequence. Some people talk about strictly monotone increasing-- and then this has to be a strict inequality.

The second thing is kind of the obvious. Parallel to this is monotone decreasing sequences. So there's these two types. And this just means that the next element, so an plus 1 is less than or equal to the previous one.

And so now there's a theorem. And I'll prove this in just a minute. And I want to come back to square root of 2. And then I want to look at another classical example of a monotone sequence.

So this is a monotone sequence. And now let me state the monotone convergence theorem. So this is the following theorem. And there's two versions of this. There's one version for monotone increasing sequence. And then there's one version for monotone decreasing sequence. I'll just state one of them, because the other one is kind of obviously a parallel.

So suppose that an is a monotone increasing sequence that is bounded. Bounded, so this just means that there exists, i.e. there exists some a, real number, so that you have a1. This says that it's monotone. This is monotone that they increase.

But the claim is that now all of the an is less than or equal to this a. Any monotone increasing sequence is clearly bounded from below because already the first element is a lower bound. So if you have a monotone increasing sequence that is bounded, then it's convergent and with limit a, where a is the sup of the an's.

So that is a monotone convergence sequence. And the main theorem about monotone convergence sequence is that if you have a monotone increasing sequence that is bounded, then it's also convergent. And likewise, if you have a monotone decreasing sequence that is bounded from below, then it is also convergent. And so maybe I will state it.

But it's just the obvious parallel. So this is part 1 of the monotone convergence sequence. Part B is that if an is a monotone decreasing sequence that is bounded from below, then an is convergent with limit a, where a is now the inf of these numbers.

Now, the first thing you might say-- and we'll come to square root of 2 in just a minute-- but you may say that this seems rather special. If you take a random sequence, obviously, it's not going to be monotone. But here's the thing. So let an be any bounded sequence. So bounded, again, means that it's bounded from above and bounded from below.

So if you take any bounded sequence, then we can form two other sequences. Obviously, we can form many others. But here's two other very natural sequences. The first one I call bn. And bn is the sup, where you're starting at an, then you take the sup of all of the ones from an and outwards in the sequence.

So this is one sequence. We know that the set of an's is bounded. So you can certainly take the sup, and you get a real number. That gives you another sequence. You can also look at cn similarly, where you take instead of the sup, you take the inf of the same set, like that.

And so in particular, just to make sure it's totally clear that if you were looking at, say, bn, so what is bn? If you're looking at, say, b3, then this is the sup where you're starting at a3, a4, et cetera. And so you see, of course, that b3 here, well, obviously, b3 is bigger or equal to b2-- sorry, no, it's not bigger or equal to, it's less than or equal to b2 because, you see, in b2, the definition of b2, so let's just write b2 down.

So b2 is when you take the sup here of a2 and then a3, a4, et cetera. But you see that b2 here-- both of them are defined to be sup-- but b2 is defined as sup over a larger set because, in addition to these elements, it also includes a2. So clearly, b2 here is bigger or equal to b3.

And in fact, the same argument tells you that if you take bn, bn is bigger or equal to bn plus 1 because this here is the sup over a larger set. It's sup over all the ones you take sup in here, plus an. So you have this property here. Yeah?

AUDIENCE: When you say that the [INAUDIBLE] sup over a larger set--

TOBIAS COLDING: Yeah.

AUDIENCE: The way that I was thinking about it is, like, well, if it said monotone increasing--

TOBIAS COLDING: If it's monotone increasing, then this here, then it becomes very special. So if the an was monotone increasing, then all of these, any elements here, would just be the a we talked about before. It would always be the limit. So it somehow wouldn't be that interesting a sequence. It would be a constant sequence. And it would be just the limit.

AUDIENCE: OK, thank you.

TOBIAS COLDING: But nevertheless, it's a good example.

AUDIENCE: But I guess the inequality still holds, then.

TOBIAS COLDING: Yeah, definitely, because it's just constant, exactly.

AUDIENCE: Yeah, of course.

TOBIAS COLDING: It's a good example. But it's just kind of almost like too simple because you get a constant sequence, right? OK, so for this sequence that we defined, so we have, again, if you take any sequence, then you have two sequences bn and cn. And bn here is now a decreasing sequence. And instead of writing decreasing sequence, then it's much easier to write it like this. This is pretty standard.

If you just look quickly at it, you know what this means. You don't have to write it out. But let's look at cn also. Remember that we assume that an could be any sequence, but it is supposed to be bounded. So this is now a monotone decreasing sequence that is bounded. So it actually has a limit. The cn's, what about the cn's?

The cn's, well, cn was defined to be the inf starting at an like that. So if you're starting like that, this means that if you take cn plus 1, then you take the infimum, but you don't include this element. So it's an infimum over fewer elements. But then this n must be larger or equal to that. So you see that the sequence cn is monotone increasing again, is bounded. And just copying from over there, the bn was decreasing.

This is an important thing. It's also something that you would see in a lot of different contexts where you define something, some sort of something, and then you have this property. And so now the thing is that if you're looking at, so a priori, it might not be so clear to compare a random element in this sequence to a random element in that. But what is clear is that if you try to compare bn with cn, it's the same set.

It's the same set. In one, you take the sup, the other you take the inf. So of course bn is bigger or equal to cn. So you have this, right? And so now you see that if you take like a random element here, if it was like further out, if it was cn, further out, I just write it as something further out. So you're adding something else to it. So this would be like further out.

We know that this sequence here-- anyway, so you get-- sorry, now you're getting this here like that. And so you have this. What I should have done is that this sequence here, so let me draw it like this. So I have that, here is maybe c1. Here is b1. And now the next element, the sequence cn is increasing and the sequence bn is decreasing.

But you always have this here. So c2 have to lie to the right of this. But it has to lie to the left of b2. So you're getting this here. So you see that this gives you a way to compare that the cn will always lie to the left of the bn's. So now what we have done here is we took any bounded sequence and we formed two monotone sequences. They are both monotone.

So this means that now from the monotone convergence that we haven't quite proven yet, but from the monotone convergence theorem, because these are bounded, we have that bn are converging. They're going down. So they're converging. So one often would write it like this more suggestively. That is converging downwards to a limit b. And the cn is converging upwards to a limit c.

And so what you get is because of all of these elements, all the cn's lie to the left of the bn's. This means that, and we'll prove this later, but it seems probably pretty obvious that the limit here, c is less than or equal to b. So we'll come back to this also next time. If I have time, I'll talk a little bit about Cauchy sequence. But I won't really prove too much about it.

But this here will play an important role when we prove the basic theorem about Cauchy sequences. The takeaway here at the moment is just, you have this concept of a monotone either increasing or decreasing sequence. If such a sequence is bounded, then it's convergent. That's a monotone convergence theorem.

And again, for any sequence, any bounded sequence, there's a very natural monotone increasing sequence and a very natural monotone decreasing sequence. Now, let's try to prove the monotone convergence theorem. And then let's get back how to think about square root of 2 in a more intuitive way using this. OK, so now the proof of the monotone convergence theorem.

So now let me just remind you what we have. So I'm only going to prove one of the two versions of it where the sequence is monotone increasing because the other one is just the same with obvious modifications. So this is the assumption. So this is assumption. So this is just that the next element is larger or equal to the previous one.

And they're all bounded by this a. Now, we set little a equal to the sup of the an. We want to show that an converts to a. Now to do that what we need is we need this stuff about that, for a given epsilon, there exists a capital N.

So now let me first make the observation that a here, since a here is the sup-- remember that the sup of a set is the smallest upper bound. So a is, in particular, an upper bound. So you have that an is less than a for all n, since a is an upper bound. It's actually the smallest upper bound. But for this you just need it's an upper bound.

I'm going to look at a minus an. And what I want to prove is that for a given epsilon, if I'm sufficiently far out, if little n is bigger than some capital N, then this here is less than epsilon. That's what I want to prove. But this thing here, you see this here, I don't need the absolute value because I already know that this is bigger than that. So I don't need the absolute value.

So this is what I want to prove. I want to prove that given epsilon greater than 0, they exists-- let me just write a little larger here. So what I want to prove is that given epsilon greater than 0, there exists a capital N such that if little n is bigger or equal to this capital N, then a minus an, which I know is positive, I just need to prove that this is less than epsilon.

So now how do we do that? Well, at the moment I've just used that a was an upper bound. Now I'm going to use that as the least upper bound. So since a is the least upper bound, then a minus epsilon is not an upper bound. a minus epsilon is not an upper bound.

So this means that there must exist-- there exists a capital N so that an is bigger or equal-- so capital N, just this one here, is bigger than a minus epsilon because otherwise, it would have been an upper bound. But it's actually strictly. Maybe I should write strictly because else would have [INAUDIBLE].

But I already know that this thing here is less than a. And also, in fact, I know more than this. I know that this thing here is less than or equal to a little n as long as little n is bigger than capital N. That was just because the sequence was increasing. And all of them is smaller than this upper bound.

So now you see, so here you have a, all the elements an lie to the left. And if you're looking at just the one from a capital N outwards, then here you have a minus epsilon. And then they lie between this one here and that. In particular, they lie to the right of this and to the left of that. So this means that they're squished in this little interval.

So this difference here, this difference from here to here is certainly bounded by the distance to here, which was epsilon. So this proves that this thing here is less than epsilon. And, well, an obvious modification-- the proof that if you take-- so this was an increasing sequence-- the proof that if you have a decreasing sequence that is bounded, that the limit of that is the inf, the proof with obvious changes is the same.

So now let's come back to square root of 2. And so we have this idea that square root 2, in a way, the way to just get an idea about how large it is, you think about it as 1 and then a number of digits afterwards. That's a kind of approximation. So let's try to make this more precise.

I'm going to define two sequences an and bn. So we'll define two sequences an and bn. And an will really be an, where an is going to be bn divided by 10 to the power n minus 1. Remember that anything, if it's 10 to the power 0, by definition, that's 1. So this here, in particular, means that a1 should be equal to b1.

And we're going to define it like this. And we are going to define the bn. And the sequences here, so there will be some properties here, that the sequence an is monotone increasing. an are just real numbers that is monotone increasing. And the an is going to be defined. And an is defined like that.

And the bn are just natural numbers. So an is defined from bn. It's really an that we are interested in. But bn is going to be a useful tool here. And the bn to some extent is very simple, because they're just natural numbers. So now we define bn. So bn is the largest natural number such that bn squared is lesser or equal to 2 times 10 to the power 2n minus 2.

So the sets of natural numbers that is bounded by that, that's like a finite set. So you just take the max or sup or whatever, the max of that set. It's just the max of that set. So that's a natural number. And now, again, we define, so now we have the bn's. And then again an's. The an are defined to be bn divided by 10 to the n minus 1.

So now the first thing you observe is that if n is equal to 1, then this thing here is 0. So this here is 1. So it's the largest natural number so that the square is less than 2. So this means that bn, so n equal to 1, bn is equal to 1. And because this thing here is also 0, so this denominator is 1. So a1 is 1. b1 is 1. And a1 has to be the same as b1.

And so that's also 1. So that's how it starts. And now we want to prove that the sequence an is monotone increasing. We want to prove that an is monotone increasing. So let's just try to understand that. And so again, for monotone increasing, you just need to compare two consecutive elements.

So this here, remember that there's ways to write a formal proof and then there's things you have on the scrap paper to get the idea. So this would be what we're going on, maybe the first statement is. You could certainly put it as part of the proof. We want to show that an is less than or equal to an plus 1.

And so this is equivalent to show that since an here is equal to bn divided by 10 to the power n minus 2, you want to prove that this thing here is less than an plus 1, which is bn plus 1 over 10 to the n plus 1 minus 1. So this here is bn plus 1 over 10 to the power n. That's what you want to prove.

So you see that we want to show, this is what we want to prove, that this thing here is less than this. If you multiply by 10 to the power n on both sides, then you get over here. And after you reduce it, you get 10 bn, so you need to show that 10 bn is less than or equal to bn plus 1. That's what we need to prove.

And again, this just comes from that you want to prove-- forget about the a now-- you can just think about that we need to prove that this thing here is less than this. Multiply over by 10 to the n on both sides, then there's still 10 times bn left. And here it's just bn plus 1. So we need to show this one here.

And so if I was to write it down, I would probably have some of this on scrap paper. And then I would instead establish this here first, and then say, therefore, and then I'll conclude why. But of course, that's not the way you're thinking about it. You first try to analyze what you have to prove. And then once you know that, then you try to write it down nicely. So we need to prove this here.

So now the thing is that we have that the definition of bn, which is right here, that's the definition of bn. But let me copy it over here. So we know that bn squared is less than or equal to, it's the largest integer so that you have this thing here. But this means that if you're looking at 10 bn, that's still an integer. And if you square this here, then this is the same as 100 times bn squared.

And so that actually has to be less than 100 times this. So it just becomes less than or equal to 2 10 to the 2n, like that. And this here you can also write as 2 to the 10 times 10 to the power 2n plus 1 minus 2. But now, you see that bn, so 10 bn is an integer and it has this property. But bn plus 1 was supposed to be the largest integer so that the square here is less than 2 times 10 to 2n plus 1 minus 2.

So you see that bn plus 1 is the largest one with this property. So you conclude that since bn plus 1 is the largest and both are integers, and bn is an integer, then bn plus 1 must be bigger or equal to bn-- must be bigger than 10 times bn, right? And that's exactly what we wanted to prove, that the bn plus 1 was bigger than bn.

So we now have that the sequence an defined this slightly cryptical way, but really, it sounds maybe cryptical, but it's just instead of using somehow digital numbers, you're just thinking about you want this and that many-- the n is referring to the number of digits, but including the digits that is before the point. So it's actually not very cryptical, but it maybe sounds a little formal here.

OK, so we have an is an increasing sequence. Now, why is an bounded? But the an is defined from the bn. So in order to show that an is bounded, you have to figure out what is a bound for the bn. But you have this here, bn is the largest such number. If I take 2 times 10 to the power n minus 1, suppose I take this number here and I square it.

This is definitely an integer. I square it. Then I'm getting 4 times 10 to the power 2n minus 2. So this in here, this here is the same as over here. But it's now 4. So this number here would not be allowed as a bn. In fact, it would be larger than bn. So bn is less than 2 to the power, 10 to the n minus 1. So this means that an, which is equal to bn over 10 to the power n minus 1, but this is now, you see, less than or equal to 2.

So the sequence an that we have defined is, as you would have expected, they are increasing because, if you think about it, it's really defined to take the number of digits, it's going to be like the number of digits in square root of 2, including the 1, the starting 1. So they all are going to be less than 2.

So we have that an is an increasing sequence and is bounded. So an, here, is convergent by the monotone convergence theorem. So the an is convergent by the monotone convergence theorem. But now, if I'm looking at an squared, an squared, well, an squared is the same as an times an.

But an, by definition, was bn divided by 10 to the power n minus 1. And so I can write it like this, 10 to the power n minus 1. And so you see that you can write this as bn squared. And then these are combined to 2n minus 2. And by assumption, if you divide by this here, this thing here is less than or equal to 2.

So you have that an converges to a. And if you want to think about it, you think about this sequence again. And so now you know that an, by the product rule for sequences, an times an, the sequence times itself, is converging to a squared. And this thing here, because each of these squares here, each of them is less than or equal to 2, then the limit must be less than or equal to 2.

What I want to prove now is that this a that we get as a limit, that the square is, in fact, equal to 2. I want to show that the square, when we're doing it, this is one more proof that square root of 2 is a real number. But this is in a way how you think about it, in terms of the decimals. And so we have this sequence an, and we have proven that it's converging to some a.

And we've shown that this a here has a property that a squared is less than or equal to 2. So I want to show that a squared is actually equal to 2. That's what I want to prove. But I've already proven that a squared is less than or equal to 2, so enough to show that a squared is bigger or equal to 2. So let's try to do that.

And I'm going to use, this is going to be another illustration of the algebraic rules for sequences. Here, now, really to prove that is less than or equal to 2, I didn't use fully the definition of the an because I didn't really use that it was the largest such that an was defined in terms of bn, where bn was the largest such number.

Suppose now I'm looking at bn and I'm adding 1 and I'm squaring it. Well, this here cannot also be smaller than this here, because then the bn should have been this in here. So this here must actually be strictly larger than 2 10 to the power 2n minus 2. And so now you just divide by this on both sides.

So you have bn plus 1 squared. And you divide it by 10 to the power 2n minus 2. So you know that this thing is bigger than 2. But you can write this in here, this denominator, 10 to the power 2n minus 2, you can write as 10 to the power n minus 1 times 10 to the power n minus 1. So in this way, you can split this up. And you can write it as bn plus 1 over 10 to the power n minus 1, and then times itself.

This is just to make sure that it's bn, and then to the bn you're adding 1. So like that, you write it like that. But you see, this thing here, if you split the fraction up, you can write it as bn over 10 to the power n minus 1. And then you can add 1 over 10 to the power n minus 1. And then you can multiply that by 2. And you can write it like that.

But now you see, now I can write this thing here. Now I can think about it in a slightly different way. So I can write this thing here. This thing here, that is the definition of an. So what I have here is that an plus 1 over 10 to the power n minus 1 times itself, but I will just write it nicely out like this, that this thing here, remember the way it goes, that this is equal to that which is bigger than 2, so this is bigger than 2.

Now, 1 over 10 to the n, you can think about that as a sequence. You can think about this as a sequence. That sequence goes to 0. This here goes to a. The sum of these two sequences, by the algebraic rules, go to the sum of the limits. So this is a plus 0, so that's a. So this thing here goes to a.

And this is the same, so this goes to a. So you see that by the product rule for the limit, the limit here is, so you get that a squared here. And since it holds for each element, then it also holds for the limit. Except you have to be a little bit careful-- it may not be strict inequality, but just big or equal to. And of course, in fact, it is equal to because now you have established the other inequality that a squared was bigger or equal to 2.

We had already proven that a squared was less than or equal to 2. So you conclude that a squared is equal to 2. If you just think about it, it sounds maybe a little bit convoluted. But it's really writing down precisely what you think about when you're thinking about a square root of 2 because you're thinking about the number of digits.

And so including the first, the 1 in 1.4, blah, blah, blah, including that 1, if you think about the number of digits, like the an, then the an is really defined to make it precise. You would define it in terms of this other sequence bn. So this was just a little illustration of how the monotone convergence theorem works. And this will play an important role next time. Yeah?

AUDIENCE: So the reason why at the top we have strictly greater than 2, and then at the bottom, we have greater or equal to 2 is because the a's and the a squared are limits, whereas on the top, we're considering each element in the sequence?

TOBIAS COLDING: So this here, the an, you're talking about this here, right?

AUDIENCE: Yeah.

TOBIAS COLDING: So the an converges to a.

AUDIENCE: Yeah.

TOBIAS COLDING: Right? I think about this here, so I think about a sequence plus another sequence. This sequence here converges to 0. So the sum of these two sequences converges to 0, and likewise for this. And then I'm thinking about it as a product of this sequence with that sequence.

But it's like each element in the product sequence is strictly bigger than 2. But it doesn't necessarily mean that the limit is strictly. But the limit has to be bigger than 2 because they could kind of approach 2. And that's exactly what happened, of course.

So I also just want to say that this limsup and liminf-- well, we didn't actually define it. And it is how you define it usually. But we haven't talked about that concept yet. But this, I talked about with that, if you take a bounded sequence, then you could form two monotone sequences. This will play a really important role for several things.

This is how one defines limsup and liminf. We haven't talked about those yet. It's also going to be used to prove an important theorem, really important theorem. And I will talk about that theorem next. But we won't prove it before next time. So what is that stuff going to be used for? And the monotone, what is another application? And in fact, there will be several other applications of monotone convergence theorem.

Here's a really important concept. And so sometimes it's hard. If it was a monotone sequence, then you could write down explicitly what the limit was. And in all of the examples we saw so far, it was easy to write down the limits. But in general, you might not be able to write down the limit, but you want to prove that the limit exists. And so this is the following notion that then becomes really handy. And that's the notion of a Cauchy sequence.

So what is a Cauchy sequence? So a Cauchy sequence, it's a sequence with the property that for all epsilon greater than 0, there exists N, just a natural number N, such that if n and m are bigger than this natural number, then the two elements an and am are less than epsilon apart. So this is a different notion than convergence. Convergence says that a sequence is convergent if everything kind of far out accumulates around the limit.

So this here says that a Cauchy sequence, that's a sequence so that if you're far out, then things bunch together, basically. Now, of course, we will prove. And so we will prove on R. We will prove the Cauchy sequence.

So we'll prove the following. Prove the following theorem-- for now, everything is in R at the moment-- that a sequence is a Cauchy sequence if and only if it is convergent. I won't prove this now, but I'll prove this next time. And again, the proof of this involved this stuff where we defined these two sequences from that bounded sequence. So it would be useful there.

Now, let me give you some examples. One example, and we'll get to that later, but at the moment, we talk, everything is in R. And then we will later generalize this. And so to generalize this, a lot of these concepts work much more general. And that's extremely useful. And it can be used to find solutions to differential equations.

But then you have to abstract these concepts to where sequences are not any more real numbers, but they are of some other space. And then you have to talk about convergence, et cetera. And so you can use this to define solutions to show that various differential equations have solutions. And so that often goes over using what's called the contracting mapping theorem. Mapping theorem.

And the contraction mapping theorem have all kinds of forms. But they're all, it's just a matter of what space you're looking at, and so on. So we're looking at R here in this case. And so the contracting mapping theorem, so a map is just a map here from R to R. And that the map is contracting means that if you take two elements in R-- sorry, T is a contracting map.

And this is really important-- c is positive. I mean, that's not important. It had to be positive. But the important point is that c has to be strictly less than 1. That's a crucial point. And then it has a property that when you take x and y in R and you're looking at how close x and y are together, so how close that the image under T of x and y, how close they are together, then those have to be some factor. And this is the factor, the c is the factor.

So they have to come closer. This is why c has to be strictly less than 1 because they have to come closer, if they come closer by a factor than the original one was. So this is a contracting map. So a contracting map is a map from R to R so that there exists some constant strictly less than 1, so that for all x and y, you have that the difference between the images is this factor that is strictly less than 1 with what they were originally.

And so now you get the contracting mapping theorem. And we'll talk about the proof next time. And again, once you know the more abstract concepts, then it's quite easy. The proof is going to be the same, et cetera, et cetera. And it will give you existence of solutions to a lot of differential equation. So this is the contracting mapping theorem.

So it's saying the following, that if T from R to R is a contracting map, then T has a fixed point. It has a fixed point. And this means that T of x is equal to x. And when you apply this theorem, so this is the contracting mapping, and again it's crucial that this c is less than 1.

That's what makes everything go. If it was equal to 1, you wouldn't get anything. It has to be strictly less than 1. And this thing here, if you set it up more generally, if the spaces are more general than just the real numbers, then this here will be that you have a solution-- it could be applied to many other things-- but it could be that you have a solution to a differential equation.

This statement here is like a standard application to get solutions to differential equations. And you can use for all kinds of other stuff. So we'll talk about that next time. We'll talk about the contracting mapping theorem. We won't talk quite yet about the applications. Any questions? OK, great.