[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: So last time, we looked at power series. So you have a sequence a n-- is a sequence. And then for each x, you form the new series-- you form the series. And the series is with an a n times x to the power n. And then you're summing-- you decide somewhere to start, and then you're summing to infinity like that.

So something like this-- that's a power series. And we looked at a particular case where the a n's-- so example here is where a n is equal to 1 over n factorial. And so we got the function E of x, which is the sum from n equal to 0 to infinity of x to the power n divided by n factorial.

And then we used this to-- we observed that E is bigger or equal to 0 for x bigger or equal to 0. And we also observed that E of 0 is equal to 1. And we defined literally to be e this function E of 1.

And then we defined another function E to the q, where q here is now a rational number. And it was defined-- in particular, E of 0 was defined to be equal to 1. E of 1 was just defined to be [INAUDIBLE]. Then we also-- we assumed, which we will prove today, that capital E of x plus y is equal to E of x times E of 1. So we assume that.

And then we defined-- so this function based on the little e-- in particular, e to some power m if m is an integer-- is just e times itself m times. And then we defined e to a power 1/n here to be the positive number, as it was defined to be a positive number so that alpha raised to the power n is equal to e. And we also defined e to a negative integer. And this was just to be defined to 1 over-- so we defined e to minus m to 1 over e to the n.

So this way, we defined-- this function here-- the power series was defined for all x. And we defined another function little e of q on all rational numbers q. And then we saw-- we assumed this here. We saw that these two functions-- so E of x is equal to e. This function here is only defined on rational numbers, so they're equal for rational numbers. So we saw that.

And then in the psets-- pset 5-- I think it's for both of them-- you will show-- Or maybe it's one for 4 and one for 5 to show that E is continuous. And you will also show that E is the-- so it would also show the following-- that if you have-- if f and g are two continuous function-- and so they are defined-- both of them-- both f and g are defined on the real numbers.

And you assume that if you restrict them to the rational numbers, then the two functions are equal. So if they're both continuous and they agree on the rational numbers, then actually, they must agree everywhere-- on all of R. So in other words, this one here-- this is a function-- by pset 5, it's a function that's continuous on all of R. It agrees with this function. And so this is the only extension of this function to be continuous and defined on all of R.

So now-- but we still needed to prove-- we still needed to prove this relationship here that we assumed. So let's try to prove this one here. And I will just do the main case of it. And so we're going to prove-- so theorem-- to prove that E of x times y is equal to E of x times E to the y. And I am going to show this proof in the case where these are non-negative numbers, both of them. The general case is-- once you have this, the general case is not very difficult, but I won't do the general case.

Now, how are we going to prove that? So to prove that, we're going to use another theorem. So the other theorem is that-- so we take two-- suppose you take two series. Each of them are of non-negative numbers.

So the setup is-- so again, this is going to go into the proof of that theorem over there. But the setup for this theorem is more general. I take a series a n, and I take another serious bn. And I'm assuming that both a n and bn for all n are non-negative, so for all n.

So these are series of non-negative numbers. And then I form another series. We will form a third series, and that third series is going to be defined from the two previous series. And the third series-- let's call it cn like this. And now cn here is going to be the sum of a i b n minus i, i equal to 0 through n.

So this has the effect that the index of this plus the index of that is always adding up to n. So the sum of the two indices is exactly n. So that's-- so obviously-- so note that the cn's here are, of course, also non-negative. They're just sums of non-negative numbers.

So this is a new series of non-negative numbers. And then the theorem is that if this series here-- if this one here and this one here are convergent, then this series here is also convergent. And we also have that this sum here-- that this sum is just the product of the two sums. So it's the product of the sum here, like that.

So again, we have these two series of non-negative numbers. From those two series, you form a third series. And the third series-- the idea is that the third series is kind of a product of the other two. And the cn's in the third series is given where you take the sum of the products of the elements in the other two, so that the index is at n.

And then if the two series you started with are convergent, then this product series is also convergent, and the sum of the product series is really the product of the sum of the other two series. Now-- so that's why we want to prove. And then once we have this we're going to use it to prove this theorem here.

And so now, to prove it, let me just first observe that-- so let me observe over here that-- so let me make a couple of definitions, and then making an observation. Let me start with making some definitions that is useful in the proof.

So I'm going to just find S n a just like we usually do where we're summing here from i equal to 0 up to n a i. The superscript comes from the a series. And then likewise, I define S n b to be the sum here of the b i's. And then I'm defining S n c to be the sum of the ci's up to n, like that.

And so of course, you have that-- we know that a i, b i, and ci are all non-negative numbers. So this means that convergence-- so convergence of any of these series-- so convergence, say, of this one is the same as saying that these numbers here are bounded That's by monotone convergence. So convergence of any of these series is equivalent to that the corresponding S n a, b, or c is bounded.

So that's just an observation. And again, this is the monotone convergence theorem. This is by monotone convergence. So that's the monotone convergence theorem. The other thing-- see, I thought of something else I wanted to note.

So let's try to prove that. So to prove it, let me just try to think about it for a second here. So if I'm looking at a1-- say, a2, plus, up to a n. Suppose I'm looking at this here. This is like S n a. And I have that S n a, and suppose I'm now looking at the S n b. This here is the S n b.

Now, if you just multiply out the product of these two, then you would get-- and I should have started at 0. Sorry. Let's start at 0 because that's what we do up there, like that. So now I multiply out, and you see that you're getting a0, b0. And you're getting-- then you can kind of-- if you try to do it-- so let's try to multiply this out, but let's do it in an ordered way

And when I'm talking about "ordered way," I'm not talking about just multiplying this one on that. You could say that's very ordered, but it's not related to the c series. So I want to just group things so that they remind us of the c series.

But if I do that-- so we're starting with this-- is this multiplied by this. Then I take this one here, multiplied by that, so a0 b1. This here is the only one where the index-- the subscripts are non-negative integers. This is the only one where the sum of the two subscripts is 0. If I'm looking at where the sum

Of the two subscripts is 1-- this is one of them, but there is another one, and that's where I take this multiplied by that. So I group those together, so a1 b0. And now I want to collect the ones-- so those are the only two where the sum of the subscripts is 1.

Then if I'm looking at, when I'm multiplying out, where the sum of the subscript is 2, well, I could start with this one, but then I have to pick b2. So I take plus a0 b2. Then I could take a1 b1. That has also sum of subscripts 2. And the last one is-- If I take a2, but then the b have to be the 0 subscript.

And now I proceed like this, so I'm grouping it like this. Now, of course, there's going to be a little bit of an issue. So for a while, everything is-- you see that this thing here-- this here is what we call c0, this here is c1, and this here is c2. And then we presumably also get c3.

But then there is a point where we run into trouble if we want to really do it like this. So what we get is that-- so when you do this multiplication, what you will see is that-- so you see like this-- a0 plus a1, a n-- what you will see-- times b0, b1, up to bn.

What you will see-- that this pattern here is continuing for a while, and you're getting that this is equal to c0, c1, plus c2, up to cn. In the cn, this is, again, where the sum of these indices could be this one here multiplied by that. The sum of the subscripts will be n.

But if you're now looking at the next one, the one that corresponds to c n plus 1, you're not getting all of the c n plus 1 because one of the c n plus 1 would be like a to the n plus 1 times b0. But you don't have that one here. So what you get is that you're getting all of these, plus something additional. And the additional is-- because everything is non-negative, the additional thing is a non-negative number.

So what we have established here is that we have-- so this has shown that S n c-- this here is-- let me just write it like this-- this here is S n c. So you see that this thing here was S n a, and this here was S n b.

So what you have is-- so this inequality just tells you that S n a times S n b is equal to S n c plus something positive-- non-negative. So in other words, this number here is less than or equal to the product of these two. So this implies that S n c is less than or equal to this here.

But now you see you already have something, because these here were increasing-- they were increasing because the a n was all non-negative. So these here are increasing, and they are less than or equal to-- the series was supposed to be convergent, so you have that-- sorry. So here I'm using that the S n a is monotone increasing up to the limit here. And likewise, the S n b here were monotone increasing up to the limit.

So this thing here-- so this is smaller than the limit. This is smaller than the limit of that. So this thing here is less than or equal to the product of these guys. And so you see-- and that's supposed to be just a real number, this here. So you see that these guys here are now bounded. So this means that-- because they're monotone. It means that that series is also convergent.

So what we have concluded here is that this series here is convergent. And we have that this infinite sum here is less than or equal to the product of the other ones. This is what this here gave us. So this is what we have so far.

Now, we want to get the reverse inequality holds. So we want the reverse inequality of what is over there so that we can say that those two things here are actually equal. Now, let's try to see how do we prove the reverse of this inequality. The reverse, we are proving by examining more closely what that product was.

So you see-- so what is this additional thing here? So again-- so we have-- we're looking at this thing here. And what we got is that this here was equal to the sum here of cn.

But then we have some additional terms. And the additional terms is-- the thing is that if you start looking at the additional terms-- so what are the additional terms? You're getting a1 bn-- that's one where the sum here is n plus 1. And then you also get-- so this is coming from this-- sorry, that one here multiplied by this. You would also get a2 b n minus 2. You get an a3 b-- sorry, not n minus 2, n minus 1-- b3 n minus 2, et cetera.

But all of these here-- and then-- so you get that, but you don't get-- again, you don't get the one that is missing here-- is a0 b n plus 1. This one is missing. And the other one that is missing is this one here. But otherwise, you get all of the ones that was in the c n plus 1, and you don't get anything more

And so this pattern continues. So the next term would be a2-- sorry-- yeah. And then you would also get something where the sum of the indices was 2. That's like a2 bn. But here, you would be missing more. You would be missing a0 b n plus 2. You're missing a1 b n plus 1. But you don't get anything more.

And so the thing is, like when you continue like this, the last one you get is when you're multiplying that by that. And that's a n times bn. Those here are the sum of the indices is 2-- 2n, sorry-- it's 2n. But you're missing a lot now. You only have one of many, many terms. But it is a term that all appear, and this one would appear in c 2n.

And so in this way, what you have is that this here is less than S to the power 2n c. They are all in there, but you will be missing terms, but all terms are non-negative So this means that you have that inequality.

So you see that now we have-- now what we have here is that we have-- what we established before was the inequality that S n c was less than S n a times S n b. But what we have now here is that this thing here, this product, is less than S 2n c.

This one here is squeezed between this one here and that one here, but both these here converge to the same. They both converge to this infinite sum. So these converge to the same. This means that the limit of this here must also be the same. And so this shows that this here is the reverse inequality. So proof-- proof over there.

So this is that if you take two series and you form the product, then that also converges if the two series is non-negative numbers. So let me just make a corollary of this, that, of course, if you take a series here-- and so I'll just do a trivial corollary, but you can see how you can use this. And this is used in proving the general case.

So we will only-- yeah, [INAUDIBLE]. So suppose I take two series here. So this is a trivial corollary of it. Suppose I take two series here, and I know that this one here-- that they are both absolutely convergent. If they are both absolutely convergent, then if you take this series here-- cn-- sorry, ci-- well, this one here I claim is also absolutely convergent.

And again, this trivial corollary is the key ingredient in proving a more general statement that would be used, say, to prove this inequality here when x and y are not-- assumed to be non-negative. But I won't get into that. But I'll just make this observation, and get this form.

So I claim that this thing here is also absolutely convergent. So again, I'm not assuming now a sign on these, but I am assuming that they're absolutely convergent. And so if they're absolutely convergent of proof, so absolutely convergent-- this just means that the series here, this one here, and this one here are convergent. That's what absolute convergence means.

But now if you're looking at ci, the ci series-- this here is the sum here, i equal to 0 to n of a i-- sorry-- a i b n minus 1. [INAUDIBLE]. But this means that ci absolute value here is less than this thing here, absolute value, like that. You can take the absolute value sign in under the summation sign. So this is less than the sum here from i equal to 0 to n, and then a i times b n minus i, like that.

But now you see that this here is the corresponding c so you could now define-- so you start with this sequence a i, and I now define a new sequence b i bar. And I do it by just taking the absolute value like that. So I know that this new series is convergent. Likewise, I defined b i bar to be the b i absolute value. And I know that this is convergent.

And then I defined a sequence here, ci bar, which is just the absolute value of the ci's. And I know that this one here-- so I know that this here is convergent. That's by assumption. This one here is convergent by assumption.

And this inequality here proves that this here-- that the c i bar is also convergent, because this one here is the one where you form the products of these two convergent sequences. So this one here is also convergent. So this here is also convergent by this theorem. Anyway, that was just an observation that we were getting this statement out of it.

Now, this is also related. So why do we have non-negative here, or why do we talk about absolute convergence? So this is a simple fact that if you take a series, if you're trying to sum infinite numbers-- we now know how to sum an infinite number.

But if you're trying to do it-- if the numbers that you're summing are all non-negative, then actually, the ordering doesn't matter. You can check the fifth element first and then the seventh and then the 119 and then come back and take 1. It doesn't matter what order you do it in.

But if the numbers are not non-negative and the series is not absolutely convergent, now it certainly matters because you could end up taking all the negative numbers first, and then you get really, really far down towards minus infinity. Then you take a few of the positive elements, and then you take more of the negative. And so then you would actually stay very far down toward minus infinity, and you would never be able to come up again.

So this is a basic fact, that if you take a series and the series are of non-negative numbers, then actually-- and it's converging, then you can rearrange the ordering, and it will still converge, and it will still converge to the same number. If you're trying to sum infinite numbers and they are not non-negative and they don't converge absolutely, then the ordering does matter.

And this is related to why we are assuming that all of these a i, b i's are non-negative numbers-- [INAUDIBLE]. But we won't get into this. You can read about it. It's a sort of standard fact. It's certainly somewhere in the book, but it's not fun.

AUDIENCE: How do we reconcile that with our typical properties, like the addition operator?

TOBIAS COLDING: Yeah, so it's related to-- again, this is not part of the syllabus, but I think it's a useful thing to keep in mind because it also explains why we're doing it in a certain way.

Suppose you take the alternating harmonic series, Remember that this thing here-- that's the harmonic series. This here is divergent. I actually proved-- I proved, or at least I outlined in class, why this series here is convergent. And this is-- and it's just-- again, the idea here is that if you take-- so you have something which is called the alternating series test.

So this is a-- I'm coming back to your question. Just a second. And this is closely related to that. So the alternating series test is that if you take a sequence a n of non-negative numbers and a n converges to 0-- let's say that a n converges to 0 in a monotone way. So an example here of this would, of course, be where a n is 1/n. That's a monotone sequence.

So if you take a monotone sequence like this-- but it doesn't need to go very fast to 0. This one does not converge to 0 very fast. But if you have that and you're now looking at the alternating series based on it, this one here is convergent. This one here is convergent.

And why is it convergent? It's convergent because of the following thing. And this is exactly what I did when I tried to explain why this here is convergent-- that you're starting somewhere, let's say starting at 0. Then you're looking at the first element.

If you're summing here-- let's say you're summing from 1 to infinity. If you're summing from 1-- then the first element here-- this is minus 1. This is non-negative, so this is a negative number. So you're going down. You're starting at 0. You're going down

The next element-- that will be positive because it's 2 here. So it's positive-- it would go up. But it's monotone, so you don't go as much up as you're going down. So you're first-- first you're going down, then you're going up. But you're going less up,e then you're going less down, then you're going less up, et cetera. And that actually is forcing it to converge.

And so now why is this related to your question about, well, how do you reconcile these? So you see, if you're looking at something like this and you now took all of the-- if you first took many, many of the positive elements and then one or two of the negative and then many of the positive elements again, then actually, you could-- for this series, if you rearrange the order of the things you're summing, and then you could either get it to diverge, or you could get it to converge to pretty much anything you want.

So this hopefully explains why when the series is not non-negative numbers, why the ordering really matters. And again, this is very closely related to this with the product of things. Why it's related? Because we're getting this remainder term. We've got something that was like the remainder, and it's the remainder that you want to be small in order to say that things are converging to the same thing. And you wouldn't have that necessarily in general unless you're assuming that the terms were non-negative.

So let's come back to-- that was just a little bit of [? side, ?] but I hope it makes things a bit more clear. Let's come back to this theorem here. How do we prove this? So we're going to prove it from the theorem that we did with that if you-- so we're going to use this theorem we just proved, which is this theorem here. So we're going to use this theorem here to prove that theorem up there.

So now it's just a matter of identifying the right series. So this is what we want to prove, this is what we want to show, and this one here-- this is the infinite sum i equal to 0 to infinity x plus y to the power n-- sorry, over-- i-- let's call it the sum from n, like that. That's what that series is.

And this one here-- that's the sum x n, n factorial, and this one here is sum y n, n factorial. Now, so if I want cn-- so the idea is that if I'm writing cn as x plus y to the power n over n factorial-- suppose I write cn like this, and I write a n x n over n factorial, and I write bn y n over n factorial.

Suppose I define them like this. I already know that this here is convergent and that here is convergent. By the way, of course, I also know that this here is convergent. But I now have that-- but I don't have-- I haven't proved-- but I claim now that this cn is really formed in this way.

So let's see. So cn here-- if I define cn to be this thing here, by the binomial formula-- so x plus y-- if you take this here to the power n, this thing here is the sum here of x i times y n minus i. And then the coefficient is-- sorry-- x i times y to the n minus y, i equal to 0 through n. And the coefficient is n choose i. That's the binomial formula.

Now, what is n choose i? Well, n choose i-- this is just n times n minus 1 down to n minus i plus 1. This gives you n elements, and then divided by i factorial.

But you can write this thing here as n factorial. If you do that, then you included too many, so you have to divide by n minus i factorial. And then, of course, you still have the i factorial in the denominator, so you can write it like that

So this means that the cn here-- our cn is-- so we defined cn like this-- x to the power plus y to the power n over n factorial. And if we use the binomial formula, then it was equal to that. So this here was equal to 1 over n factorial sum i equal to 0 to n x i y n minus i, and then n choose i-- like that.

Now I write in this little equation here. And so I get that 1 over n factorial summing from i equal to 0 through n, and then n factorial, n minus i factorial times i factorial, x to the i times y to the n minus i.

And now you see that the n factorials here cancel with the n factorials. So you're left with i equal to 0 through n. And now I can group it. I can group it. I can write it as x i divided by i factorial times y over n minus i factorial. And this is to the power n minus i, like that.

But now you see that this here is exactly when we defined a n like this and bn like this. This here is exactly the sum from i equal to 0 through n of a i b n minus i. And so you see that what we have is that this thing here-- so E of x plus y-- with our definition of the c's, this was just defined to be equal to this limit.

But the cn's is also given by the product of these two other series. But this means that it must be the product of the two, like that. It's the product of the two sums, infinite sums, like that. But this thing here is, of course, just E x, and this is just E y. So this proved it. And again, this here was just assuming that the x and y are non-negative. But given what I said-- also, this corollary-- it's not very hard to prove the general case from that.

So now-- but I won't do that. So let's just talk a little bit more about continuous functions. So let me one more time just remind you about it, and then let me give you a wild example of a function.

So first let me remind you with continuous. So continuous [INAUDIBLE] is a really important concept, also, in this class. So I have a function, and it doesn't need to be defined. It's defined, say A on some subset of R into R.

And then we say that-- and then we have some point x0 in the domain. And we say that f here is continuous at x0 if the following hold. That's for all epsilon. For all epsilon greater than 0, there exists delta greater than 0 such that if x is in the domain and x minus x0 is less than delta, then the images deviate at most by epsilon.

So this is, again, what it says to be continuous. And here, unlike the previous couple of times when I defined it, here it's really defined. You see it makes total sense on any subset of R. It doesn't need to be an interval, even. So it doesn't need to be all of R, but it does need to be an interval.

Now, so this is what it means for a function to be continuous. Let me give you an example of a sort of somewhat wild function. And so continuous is a very nice property. It's continuous if it's continuous at all points. Usually, continuous is a nice property to have. Often, it's the bare minimum. It's often not enough. You want something more than continuous. But of course, other times, you may have much less.

So here's an example. So this is an example. So it's a function defined on all of R-- example of a function defined on all of R that is nowhere continuous. By "nowhere continuous," I mean that there's not a single point where it's continuous.

And so this is the function f of x. And the function is-- if x here is a rational point, then f of x is 0. And if x here is an irrational point, then [? I'm ?] setting it equal to 1. So it's a pretty simple function.

And so I claim that this is not continuous at any point. And so the thing is that-- you see, if you take any points-- let's just take a rational point first. So suppose you take a rational point here. This here is-- x0 is rational.

Then you have that-- if you are looking at square root of 2, we know that this here is not rational. We proved that. We showed that this is not rational. And so in fact, if you take 1 over-- sorry, square root of 2 divided by n, where n here is sufficiently large, then you can make this thing here very close to 0.

And then you could add this thing here. So you could look at x0, and you could look at square root of 2 over n for n really large. If n is really large, you can make this thing here, so this is square root of 2 plus x0. So you could make this really, really close to x0. This thing here would be irrational, and so the value of the function here would be 1.

Let's make this a little bit more precise. And so-- let me maybe say it just before I make it more precise. So let's just draw the graph here. So we have here x0. We have here this-- this here is this irrational number, which you could think about, but it doesn't need to be this number. It could be anything that is irrational.

This one here is irrational. And by making n super large, this here super close to x0, the value of the function here is 0. The value of the function here is 1. And so you see that you have these points as close to this point here as you want, but the value of the function is very different

So again, just to spell it out, given an epsilon-- so you see that if epsilon here was smaller than 1, then you would have that f of x minus f of x0-- this here is a rational point. So this here is-- this is 0, so this is just f of x. And if this here is irrational-- so if x here is irrational, then this here is 1.

So you would have that you can choose this irrational point. So for any delta, we can choose x, which is irrational, and so that x minus x0 is smaller than delta, but we have that f of x here is actually equal to 1.

But this difference here was supposed to be smaller than delta. That is not the case. If epsilon is smaller than 1, it's definitely not. It's 1. And so similarly, if you take-- so this was proving-- so this showed-- so this proof proved that f is discontinuous at all rational points.

Now let's prove that it's discontinuous at all irrational points. So if x [INAUDIBLE] 0 is irrational, then we have, of course, that f of x0 is equal to 1. But we also know that arbitrary close-- so for any delta greater than 0, there exists a rational point, a rational number x, so that x minus x0 is smaller than delta.

But now, again, we have that f of x minus f of x0-- this thing here is 1, this thing here is 0, so this difference here is 1. And so you couldn't make it strictly less than 1 by having this delta sufficiently small. So this proof is sort of same thing. It proves that the function is also discontinuous at all irrational points.

Now, let's talk a little bit more-- so let's talk a little bit more about some of the algebraic rules that we discussed last time. And I wanted to single out two of them.

So we had the following. If f and g are continuous functions and c is a constant-- then last time, we talked about the following. We talked about that f plus g, the sum of these two functions, is also continuous.

We talked about that if you take a function and you multiply it by a constant, this is also continuous. We talked about that if you're looking at the product of these functions, that's also continuous.

The ones we mentioned, but we didn't actually-- this here was so elementary, this one here I outlined, and let's talk about the last two properties. One is close to what we had for sequences. And that was that if f here does not vanish, it's nowhere vanishing, then 1/f here is also continuous.

And the last one, which is sort of different from anything we had for our sequences, is that if you have now a function-- so you have these two functions, f and g. But now I'm thinking about f here as going from, say, some subset of R into R, but it may not [INAUDIBLE] into some set B. This is a subset of R. This is another subset of R.

And then I have g here goes from this set into R. It needs to be-- so you need this here to be defined on the images of f. If you have that, and these two functions are continuous, then if you form the composition-- so this is g of f of x-- this new function h here is also continuous.

And so I want to-- in the few remaining minutes, I want to talk about these two properties. And I will start with the last one, because that's somewhat different from stuff we talked about last time-- I mean for sequences.

So let's try to prove that this composition here is continuous, so proof of 5. So given epsilon, well, we know that the function-- you sort of to start backwards.

Given epsilon, you know-- sorry. Let me just say that I want to prove it. Let's show that h, which is the composition, is continuous at x0. And so I want to prove that. And for convenience, I'm just going to call the image of x0 under f, and we'll call that y. So now, given epsilon bigger than 0, since g is continuous at y0-- note that this is y0.

So since g is continuous at y0, there exists a delta-- let's call it delta g-- so that if y minus y0-- if this is less than delta g-- then the images-- sorry, under g not under f, but under g. So then the images g of y minus g of y0-- the images under g is smaller than epsilon. So this is just because g is continuous, and I'm using that it's continuous at y0.

Now I need to use that f is continuous. So now I'm thinking about this delta g as my epsilon [INAUDIBLE] in f. So from delta g and the continuity of f, there exists delta of f such that, if x minus x0 is less than delta of f, then the images f of x minus f of x0-- the images is less than delta g.

But now you see-- now I have-- this thing here-- f of x-- this thing here-- this is f of x. And this thing here, f of x0, was y0. So this is the same as this. But now you see that-- so I have these two points that they are-- and where one of them is y0, and then I have this other point, and their difference is bounded by delta g.

So I'm in iteration. So this means that the images of these two-- so this implies that the images of this thing here under g and this here on g-- so g of f of x minus g of f-- sorry, g of y0-- this thing here is less than epsilon. This is just because that was how delta g was chosen. And you see that one of them-- one of these two points have to be y0, but that's exactly what it is over there.

So I have this-- but you see, this thing here-- but this thing here is just-- but you see that g of f of x-- this is just the composition. So this is what we call h of x. g of y0, where y0 was defined to be f of x0-- so this thing here is g of f of x0. But that's just the composition at x0.

So you see, what we have now is-- So therefore, we have that if x minus x0-- if this thing here is less than this delta f, then we have this thing here. But this thing here is h of x minus h of x0 is less than epsilon. And so this proves that the composition of these two functions is continuous.

And so again, when you have two functions and you look at the composition, it's really important to first check that the second one is defined. So the second one should be defined where you end up under the first map.

So now let me just say a few words about the other one, number 4. But the number 4 is closer to things we had for sequences. So we have now proven [? 4. ?]

So what we have is we have this function, it's never 0, and we have some x0 in the domain, and we know that f here is continuous. We only need it to be continuous at x0. And then I want to show that 1/f of x is continuous at x0.

And so I'm looking at-- so I want to estimate this difference here. And I want to say that if I pick an epsilon, then I can pick x as close-- if I pick x to be within delta, where delta is sufficiently small, of x0, then this difference here can be smaller than epsilon.

So now, let me just estimate this thing here. And so this thing here is just-- so now I just put it on the common denominator. So I have f of x0 minus f of x divided by f of x0 f of x [INAUDIBLE].

And now it's sort of the usual problem, but now we're maybe not so worried about that usual problem. The usual problem is this x0 is fixed, so this is just some number. So that doesn't really matter too much. This one here is not fixed, so that's the usual problem. This is a problem we've seen before. We saw it in 3 up there, and we saw it in the similar things for sequences.

But now the thing is that since f of x0 is not equal to 0, then you can just take-- so then now you can first take-- so now we're going to pick the delta we want in two steps.

The first step is since this here is not 0, then we know that there exists a delta 1 greater than 0 so that if x minus x0 is smaller than delta 1, then the image between f of x and f of x0-- I can make that smaller than f of x0 over 2. You see, I'm just using this one here first as my epsilon.

Once I have that-- but this here implies, of course, that f of x-- see, I have f of x in the denominator, and I want to bound this here from above. I want to bound it from above. So I just need to get a lower bound for f of x. So f of x here-- so I have that f of x plus-- I can write-- sorry, I can write. Using this, I can write f of x0 as f of x minus-- sorry, f of x0 like this-- minus f of x plus f of x. Obviously, I can write it like that.

And so this means that f of x0, slapping an absolute value sign on it-- this here is less than f of x0 minus f of x plus f of x. And so now I can move this one here over on the other side. So I have that-- so if I move this one here over on the other side, I get that f of x absolute value is bigger or equal to f of x0, absolute value, minus this difference here, like that.

But if I make x and x0 delta 1 close to each other, then this difference here is at-- it's actually less than this thing here. So there's still another 1/2. You see, this one here must be bigger or equal to 1/2 of f of x here, because this here takes up strictly less than half of this thing here, so there's still another 1/2 left

And so this gives us a lower bound for this. And now we are out of time. but this gives us a lower bound for this, as long as we make sure that x and x0 is delta 1 close. And once you have that, you can just treat this here as a constant. And so now it's clear that you then make another choice of delta, possibly even smaller than this one here. And then for this smallest of these two numbers, then you can now conclude that this fraction here is smaller than epsilon. Any questions? Yeah?

AUDIENCE: I had a question going back to what we were talking about [INAUDIBLE]. I was a little bit confused because we're defining it to be non-continuous. But by definition of continuity, if you picked two x's near each other that were both rational, and their f of x is both 0, then wouldn't that be continuous?

TOBIAS COLDING: So first of all, let me say that if you were to picture this function-- of course, it wouldn't be very easy to picture it. But you would sort of-- if you were to draw it, then actually, it would look-- it wouldn't look like a function anymore, because you would have it on all the rational numbers here, it would be 0, and all the irrational numbers here, it would be 1. So if you picture it, it would look like two lines. But of course, it wouldn't be two lines because it's-- but the points are just so close together that you couldn't really draw it.

And so the thing is that-- but it also means that if you-- but this picture here illustrates what you're saying, that for-- if you take rational numbers, well, all the rational numbers that is really-- any rational number, whether it's close or not-- it will all be 0 on those.

So that's not where the problem with continuity occurs. The problem occurs because you also have irrational numbers close to it. And it has to be for all points close to this that the value has to be almost the same. So it's really for the irrational numbers that the problem occurs with continuity-- in irrational numbers. And vice versa for an irrational number-- if you're looking at continuity around an irrational, the problem is not from other irrational numbers, but is from the rational numbers.