[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: OK. So first, let me look at a curve and plane. Talked a little bit about this last time at the very end. Oops. OK. So let's first talk about curves in the plane. So we have functions, f and g, on some interval, say, from a to b into R. And I'm assuming that f and g are differentiable and that the derivative are continuous.

And then I'm thinking about-- so I'm thinking about the curves. So gamma here of s is a curve. And it's given by the gamma of s is f of s comma g of s. And I want to look at the length of this curve. So s here is running in the interval from a to b.

And so the length-- and this is the definition of the length, but then we will justify it. So the length of gamma. So L is equal to the integral from a to b of square root of f prime of s squared plus g prime of s squared ds.

Now, why is this the case? So why is this a sensible definition? So suppose that this thing here is f of a comma g of a. So this is the starting point of the curve. And suppose that the curve looks like this. So here is the endpoint. And now imagine that you take the interval from a to b and you partition it into a lot of small-- so you have this partition of this interval.

And then you're looking at-- so we have the curve here. This is maybe xi minus 1. And here I have xi. And under the map gamma, this little interval here is mapped maybe into this piece of the curve.

Now, this piece of the curve, I will then approximate it by the line segment from here to here. So if you do that, then it means that you are looking at-- so you have that you're looking-- so you have that-- so here you have xi minus 1. Here you have xi. And you're looking at the curve on this little interval.

And there the curve goes from-- so the initial point is this. That's the image of that point. And the image of this is gamma of xi. Wow. And so the line segment from here to here-- so the length of this line segment if you have this point here-- so this is gamma of xi minus 1, and this is gamma of xi.

Then the length here is the vector here. So gamma of xi minus gamma of xi minus 1. The length of this. But this thing here is-- this is the point. Sorry. Yeah. And the beginning point is this. So this here-- so it's the length of this. But the length of this by Pythagoras is the square of the coordinates. And then taking the square root of that. So this is what the length of the line segment here is.

But now, the mean value theorem tells you that this thing here is essentially the derivative at some point in between these two. It actually is equal to, if you will, the derivative at some intermediate point times the length of that little interval. This is this difference here.

And likewise-- so this is x, like this, some intermediate point. And likewise for this, there's some intermediate point times the length here. So this means that this little line segment here, the length of this little line segment is equal to-- and you see that when you take this thing here and you square it, then you can factor out.

So let me just write it out. So you take this thing here. So it's f prime, like this, squared, delta xi squared plus, and then g prime squared delta xi squared. But now you can factor this out. So you can write this as square root of delta xi squared, and then f prime, like this, squared plus g prime in this intermediate point squared.

And this here factors out both of them. And so now you can just-- when you take the square root, it actually goes out as-- this is anyway positive. So it becomes like this. And now the functions g prime and f prime are both continuous. So this thing here is roughly what it is at one of the endpoints. So this here is roughly equal to, like this.

And then it's like one endpoint. Or you can even take the-- yeah. So it's like an endpoint. And so you see that this thing here now, when you sum up-- when you sum up all of these over this partition and you let the partition-- the delta xi goes to 0, then this here is going to converge to the integral from a to b of the square root of f prime of s squared plus g prime of s squared ds.

So that's why. So you take this little curve. It's parameterized. So you take the curve parameterized by a to b. You're looking at a partition. And then, on each of these little intervals, you approximate the curve by the line segment. And then you let the-- as you let this refinement-- the length of these individual intervals goes to 0, then actually this approximation will have a limit. And that limit is this integral. So that's why we define it to be the arc length.

OK. So that's arc length. And so let's look at an example of this. So we'll look at two examples. Let me first look at the case where f of s is equal to s and g of s is equal to s squared. Then I have that. So then the curve is-- so the curve gamma of s is s comma s squared. And f prime of s is equal to 1 and g prime of s is equal to 2s.

And so this means that-- and let's say that it's parameterized, s here, from 0 to 1. So then the length here of this curve is the integral from 0 to 1 and then square root. And then you have to take this thing here squared. So that's 1 plus this here, 2s squared, ds. So this is-- and it's this integral here, 1 plus 4s squared. So that's the arc length. So it's given us this integral.

The second example I want to look at for arc length is if you have the unit circle. So suppose that you have the unit circle. So here you have one. So this is supposed to be the unit circle. So it's x squared plus y squared equal to 1. It's a unit circle centered at the origin.

Now, this means that if you take a point, then y here is-- so this here means that y is given in terms of x, where that y squared is equal to 1 minus x squared. And so if you assuming that you are in this upper, where y is positive-- so if y here is positive-- then you're looking at y equal to square root of 1 minus x squared.

So in this part here, where y is positive, you have the curve-- this part here of the unit circle. This one here is given by the parametrization gamma of s equal to s, and then square root of 1 minus s squared. And s here is now running-- s is the first coordinate. So it's between minus 1 and 1. So s here is between minus 1 and 1.

OK. Now suppose that you take some x here. And suppose that you want to look at, what is the arc length of this part of the circle? Well, then, in this case, if you're looking at the arc length of this part of the circle-- let's call that arc length theta. So theta here is equal to this part. It's not parameterized like this. But x here is-- so this here is-- so length of gamma.

But now gamma-- by now, s here is running from x to 1. Because x is the first coordinate and it's running from-- the first coordinate is running from x, from this point, this here, to 1. And so now, let's try to calculate this length, or at least write it up as an integral.

So if we write it up as an integral, we have that-- so in this case-- so gamma s is equal to s square root of 1 minus s squared. And so if we define f of s equal to s and g of s equal to square root of 1 minus s squared, then we take f prime-- well, that's equal to 1-- and g prime.

So then we use the chain rule. So this is equal to 1 over 2 square root of 1 minus s squared. And then we have to take the derivative of this also. So this is times minus 2s, like that. And so this here is, of course, then equal to. The 2 cancels. And you're getting minus s over square root of 1 minus s squared.

And so now, if you're looking at-- so f prime squared plus g prime squared, this is equal to f is 1. So this is 1 plus this thing here squared. So this is s squared. And then the denominator squared is just, of course, 1 minus s squared. And we can now put it with a common denominator.

And so you see that this is 1 minus s squared. 1 minus s squared. That's how you write 1. And then plus s squared 1 minus s squared. And so you see that this whole thing is the s squared cancel in the denominator. And so you're getting 1 over 1 minus s squared.

And so this means that the integral-- so this means that the integral-- so this length here-- and the length is-- so actually, we call the length theta. And it will be clear in a minute why I do this. So the length here is then the integral from x to 1. And then it is of the square root. So it's supposed to be the square root of this thing here. So it's the square root. So if you will, it's 1 square root of 1 minus s squared ds.

So that's the length. But if you think about it-- so you have here the unit circle again. What we now calculated was that we take some x here, and then we calculate the length of this piece here. Well, that piece is-- this here is also-- this is how you define. If you will, this is the angle. So theta is the angle.

And so you see that-- so theta here is the angle. And x here-- x here is-- and this point here-- sorry, this point here is cosine to the angle comma sine to the angle. That's this piece. So this means that x is actually cosine through the angle.

So this means that theta here-- in terms of x, theta is the inverse function. So it's what's called arc cosine to x. So this arccosine, it means that it's the inverse function. If you take the inverse function on both sides of this then-- so if you take the inverse function-- so arccosine. So you get arccosine to x on this side. This side, you're getting arccosine and then cosine to theta. But these two functions together gives you the identity. So this is just theta. OK.

OK. So in other words, what you have is that arccosine to x is given as the integral from x to 1 1 over square root of 1 minus s squared ds. So you have this formula here, which is not so surprising. But anyway, you can get it from this. And so now you can get it-- so now, the fundamental theorem of calculus.

So by the fundamental theorem of calculus, then you have that. So if you have a function that is given like an integral like this, then this thing here is the derivative. That's what the fundamental theorem of calculus say. Except you have to be a little bit careful because this is if the variable is up here. But now it's down here.

Because it's down here, the derivative is minus this. OK. Let me explain that. So you get from the fundamental theorem of calculus, the derivative of this function, I claim, is minus this thing here. And then this should be evaluated at x. This is the arccosine at x is equal to that.

So now let's just see that. So why is this the case? So you take-- so remember that if you have a function f of x and this is given as you're integrating from some a to some x, a is fixed. And then you're integrating a function like this. Assuming that the function is nice, et cetera, then we have that the fundamental theorem of calculus-- fundamental theorem of calculus-- the derivative of this function is little f.

But suppose I now switch. So now suppose I now look-- so I don't look at this, but I look at-- suppose that g of x is the integral from x to a-- let's call it x to b, just because we usually have the endpoint as the last point, as b. So suppose that the function is like this. What is the derivative?

But this is just the fundamental theorem. But one of the rules we had writing the integral is we can write this one here. So here I'm thinking about that a function little f is defined on some interval from a to b. And I can rewrite this thing here in this form here because I can just think about this thing here as the integral from a to b of f of s. So sorry. Let me write it over here.

So I'm just making the trivial observation that we have-- so if we have a function from a to b into R, and we have some x that lie between a and b, then we proved that the integral from a to b of f, this thing here, is equal to the integral from a to x plus the integral from x to b. That was one of the rules, that if you take this-- if you have a dividing point in this interval, then you could integrate over the first half and then add the integral of the second half.

But this thing here just tells you that this thing here you can rewrite as this minus that. So you have that the integral from x to b f of s ds is equal to the integral from a to b f of s ds minus the integral from a to x f of s ds. And so a and b are fixed. And when you think about this thing here as a function of x, then it's equal to-- this is just a constant. And then minus this.

So when you take the derivative of this, this is a constant. So this derivative is 0. The derivative of this, well, that is straight the fundamental theorem of calculus. And because of this sign, you get minus f. And so that's what I applied over here, that if it is of this form here, then the derivative of this function is minus little f. OK?

OK. So this was-- this is so-- so this is how the angle is defined, if you will. The angle is defined as the arc length of the curve. And we can calculate it using this formula here. And we're getting a nice-- I mean, we know what this arccosine-- we know actually what the derivative is just from this arc length and the fundamental theorem of calculus. OK.

The next thing I want to talk about is I want to talk about pointwise and uniform convergence of functions. So you have a sequence of functions. You have a sequence of functions fn. So suppose you have some interval I, and you have a sequence of functions fn. And you have another function f also on this interval.

Then we say-- so this is the definition. We say that the sequence here converges to f pointwise if, for all x fixed-- if you're looking at this-- so x here, think about x as being-- x could be anything, but it is fixed. And then you're looking at this sequence here. So this is a sequence of real numbers because x is fixed. Then you want this thing here to converge to f of x. Yeah?

AUDIENCE: Sorry. I had a quick question about the last thing. So how do you know that the integral of a to b fs ds is 0?

TOBIAS COLDING: So no. So the thing is that what we're looking at here-- so I was asking-- so this here is-- so if you have a function, capital F, that is given like this, the fundamental theorem of calculus gives you what the derivative is. So I'm just asking, suppose that the function-- that the variable was not up here but it was down here. What is the derivative?

And so in order to figure out that derivative, I'm just using that-- so if I was integrating over the whole interval, a and b are fixed. So this is just some number, whatever it is. But it's probably not 0. I mean, it generally wouldn't be 0, but it's just some constant. And so for each x you get the same constant here. And then it's equal to this plus that.

But this is what we're interested in. So I'm just moving this 1 over on the other side. So I get this thing here. But remember, this here is a constant. So when you take the derivative, it's 0. OK. So--

AUDIENCE: For this problem--

TOBIAS COLDING: Yeah?

AUDIENCE: --if you were to just substitute-- set b equal to a, then wouldn't you just get that a equal integral from a to a goes to 0? And then you can get this--

TOBIAS COLDING: But that's-- but if-- right. If b is equal to a-- if b is equal to a, yeah. I mean, but it's not. I mean, so then this interval is just a trivial interval.

AUDIENCE: Right. But doesn't it allow you to get to the same identity there?

TOBIAS COLDING: Right. But the thing is-- so the reason why-- so why did I want to do it? Yes. But it's like-- but it's kind of using something implicitly. You see what I mean? I'm assuming that x is between a and b because when we took this interval, if x is not between a, b, we have to figure out, what the hell do we mean by this sort of integral anyway? And so, yes, it works, even if you have the ordering differently, but you just have to-- all I want to point out here is that if it is downstairs and not upstairs, you have to just remember the sign. OK.

OK. So this here-- so I have a sequence of functions on the interval. I have another function on the same interval. And I will say that this sequence is converging pointwise to f if for all x fixed, when we're looking at this sequence, it converges to that. And so let me look at two examples of that.

So the first one is-- so the first example I want to look at, this is where the functions fn of x is equal to x to the n. And x here is between 0 and 1. And so this is my fn and my function f, which is now, of course, also a function on this interval. This is equal to 0 if x is bigger or equal to 0 but strictly less than 1. And it's 1 if x is equal to 1.

OK. So of course, this function f is not continuous. The other ones are continuous. Now, I claim that this sequence-- so the sequence fn is converging pointwise to f. So fn converges to f pointwise. Because if you fix x-- let's say, fix x, and let's assume that it's between-- sorry, it's between 0-- it's strictly less than 1.

So you're fixing x. And let's first deal with the case where that x is strictly less than 1. Well, if I'm looking at x to the n-- so it takes some small number. Small just means that it's strictly less than 1. And I multiply it by itself n time. Well, that certainly goes to 0. If x here is fixed but it's not 1, then x to the n. If x is 1, then this here is still 1.

So you see that in this case-- so this here is f to the n of x. And this here is f of x. So these things here, I mean, they're actually equal to this. So in particular, they converge. And this here is what f of x is. And this here is, again, fn of x. So you have that this sequence of functions converge to this slightly crazy function.

So that's the first example. They converge pointwise. The next example I want to look at is-- so the next example is where the functions are-- you're just summing here. You're summing. You just take the power series for the exponential function.

So you're summing where k is equal to 1 to n. And that's my fn of x. And then my f of x is the infinite sum, which is the exponential function. And we have already proved that if you fix x-- so we fix x-- then this fn here of x converges to f of x for each fixed x. For each fixed x.

So this sequence of functions pointwise converges to the exponential function. So this is one notion of convergence for functions. And there's another notion that is also useful. They're both useful. The other one is stronger. And so of course, if you have the stronger version, you often get something much better. Oh, I shouldn't have erased this. But anyway.

And so the second notion is-- so this is the notion of uniform convergence. Uniform means that it's uniform in x. So what we have here is we have a sequence of functions and f defined on some interval into R. And then we say that this sequence converges to f uniformly if, for all epsilon greater than 0, there exists a capital N so that if little n is bigger than this capital N, then f of x minus fn of x is smaller than epsilon for all x.

And again, the uniformness, it holds for all x. You can use the same capital N for all x. And so now, now what you have is that if a sequence converges uniformly, then it converges pointwise. So we have the following lemma that follows directly from the definition. So lemma.

So a uniform convergence of fn to f implies pointwise convergence of fn to f. OK. So uniform convergence implies pointwise convergence. And that's clear. That's really just the definition if I had written out what it means for-- so this here, I've written out what it means to have uniform convergence.

If it was pointwise convergence, then you have these functions again. And you have that this converges to that pointwise. It means that the sequence fn-- so once you fix x, then the sequence fn of x-- so once x is fixed, then these here converge to this.

But saying that this sequence here converge means that for all epsilon greater than 0, there exists an N. But this N is now allowed to depend on x so that if little n is bigger than this capital N, then fn of x minus f of x is smaller than epsilon. So you see that clearly uniform convergence implies pointwise. And again, uniform just means that this capital N can be chosen to be the same for all x.

So now let's look at the examples. Unfortunately, I erased the most important one. So I will just go through it again. Let me just talk about the example again. And then we'll see why this example here does not. So we have this example from before, where the sequence of functions was x to the n. And x here was between 0 and 1.

And then we had this function f that was this function that was not continuous. But this function was given to be equal to 0 if x here is strictly less than 1 and it was 1 if x is equal to 1. That was this function f. And we know-- we've already shown that the fn converges to f pointwise. But we want to see now-- so I claim that fn does not converge. So this does not converge uniformly to f.

We know already that it converges pointwise. And I claim that this sequence does not converge uniformly. So now, why is this the case? Well, this is because, you see, by the intermediate value theorem, the functions fn are continuous. At 0, they take the value 0. At 1, they take the value 1. So for each n, by the intermediate value theorem-- since each fn is continuous, then there exists an xn, but it really will depend on n. So that fn of xn will take the value one half.

So the functions here-- this is the functions-- oops. This is the functions. This is the function fn. At 0, it takes a value of 0. At 1, it takes a value 1. So there is somewhere where a half is achieved. And this is at this-- it's here. So you have that. And again, this xn is depending on n. And the thing is that as n gets large, this point will have to move very close to 1.

But if you're now looking at fn of x minus f of x-- so we want to-- if it was converging uniformly, it would mean that this difference here would be smaller than some epsilon for all x. But if you're just looking at this point here, this difference here is this point here, xn, will lie strictly-- xn here will lie-- this is where this value here is a half. It's not 1. So it's strictly between 0 and 1.

So you see that for each n, this function f at xn, this will always be 0. This is because the function f is 0 except at 1. And this point is definitely not 1. So you have that. So you see that this thing here, this here is 0. So this is just the value of this, the absolute value. But that's actually a half. So this is always a half.

So the function would converge uniformly as long as-- so uniform convergence would mean-- so uniform convergence. So now let's show. So let's prove that fn does not converge uniformly to f. And so I just pick my epsilon.

So I pick an epsilon less than half. And then if it was converging-- and suppose not-- that is, suppose fn converges to f uniformly. This means that for all epsilon. But I just use this epsilon here. I just use this one, epsilon.

But then I would have that there exists capital N so that-- I mean, I can even take it to be equal to a half. So there exists capital N such that if little n is bigger than capital N, then this thing here, minus f of x, would have to be strictly less than half for all x.

If it was converging uniformly, then in particular, for this one choice of epsilon, you would have this for all x. But we already know that if we are looking at-- now depending on what this is, but that's fine. If you insert x equal to xn, then this is actually equal to a half. So this here does not hold. This is the contradiction.

Now, in the other example that we looked at before-- in the other example we have that-- in the other example, which is this example here, this sequence here is converging. And I'll return to this in a minute. So in this case, these here converge to f uniformly on each interval-- on each bounded interval. So each interval of the form minus L to L. So on each bounded interval. OK. And we will talk about this more in just a second.

OK. And because this is the content of what's called Weierstrass M-test. So this holds more generally that if you take a sequence-- so let's see. Weierstrass M-test. So in Weierstrass M-test, you have a sequence of functions, fn, and you have that these sequence-- so fn, sorry, is functions on some interval into R.

And then you have that the fn are bounded by some constant Mn. And then you have that the Mn-- so these are clearly non-negative numbers. And you have that this series here-- this here is finite. So this is a series of non-negative numbers. So to say that this series is summable is the same as saying that it's finite.

And now the claim is-- so then Weierstrass M-test is-- so this is that if this here is the case, then this sequence converges to some f uniformly. That's Weierstrass M-test. Yeah?

AUDIENCE: What does this mean here? Is it the absolute value of the function?

TOBIAS COLDING: Oh, yeah. So this here-- so this here means-- this here just means that fn of x is less than or equal to M for all x. Yeah. That's a kind of quite common used situation. OK. So let's try to prove that.

So we are looking at-- so suppose we look at-- suppose we define Sn of x. So this is just when you're summing from k equal to 0 to n of fk of x. Suppose you're looking at this. And you want to prove-- so I want to show-- yeah, sorry about that. This is not what we want to prove. Sorry.

Sorry, I should have said. Yeah. It's not these functions that converge. It's the sum of them. So what I want to prove is that-- I want to prove that this thing here-- as k equal to 0 to n, that this function here converge uniformly. OK? So think about it. You want this to probably apply to some power series. And so the individual fk is like this polynomial. It's of some degree. OK.

So we define this function Sn. And I want to prove this Sn and then converge. And if it's OK, I think it's better to call this something else than little f, to call it S, because it's not these guys that converge. It's the sum of them that converge. So we want to show that these guys here, the Sn converts to this S uniform. That's what we want to prove.

Now, let's look at the difference between Sn of x and Sm of x. And so here I'm going to think about n as being larger than m. Suppose I'm thinking about this thing here. Well, this here is when you're summing the first-- I guess you're starting at k equal to 0. So then we're summing the first n plus 1 of those guys.

Here, we're summing more. We're summing those guys, but then we're also summing some more. And so this thing here-- so let me just write it out. So k equal to 0 to n of fk of x. And the second one is when we're summing. n is bigger than m. So here, we are actually summing fewer.

And so of course, the difference is between these two are, how many more of these guys are there than those guys? So this is summing absolute value, where we're summing from k equal to m plus 1 up to n of fk of x. So that's what it's equal to. And we didn't even have to have absolute value. But in a minute, it's useful to have absolute value.

Because now, of course, this sum here is less than the sum of the absolute values k equal to m plus 1 to n. And then we take the absolute value in under the summation sign, like this. But now we can use our condition here. So this thing here is less than or equal to the sum here from k equal to m plus 1 up to n of Mk.

But those guys, this is just a sequence of real numbers. This is not a function. It's just real numbers. And they converge. So this means that-- so this means that given-- this means that you have-- that you can make this thing here as small as you want, as long as m is sufficiently large.

So given-- so given-- so given epsilon greater than 0, there exists a capital N such that if m here is bigger than capital N, then the sum here of the tail, this thing here, up to anything else of this Mk, that this is less than epsilon.

And so you see that this is really saying-- so this implies that the Sn for each x-- for each x, you can think about this thing here as a sequence. And so you have that now this sequence here is a Cauchy sequence. For each x, this here is a Cauchy sequence. So this means that this Sn of x, they will actually converge to something. And that's something I call S of x. So that gives you a function S of x.

But it gives you more because you see, it actually shows you something about-- because this inequality here, that this difference here is less than that. That holds for all x. So this means that-- this means that as you're sending-- so this means that you have-- so now let n go to infinity. If you take this inequality here-- so you we have this inequality, Sn of x minus Sm of x is less than or equal to the sum here, k equal to n plus 1 to n of Mk.

So we have this inequality. And in fact, you can write it as if you're summing over the entirety. And so now this side of it, this side does not depend on little n. This here holds as long as all here we require was that this here was bigger than that.

And so as n goes to infinity, you have that S of x minus Sm of x will have to be less than or equal to the sum. And the key point here for this Weierstrass M-test is that this here holds for all. This inequality holds for all x. You see that little n is gone. There's only this capital N. Sorry, the M.

And so now we again-- so as long as you choose some N insufficiently large, then if little m is bigger or equal to this N, then you can make this sum here that have no reference to x. You can make this less than epsilon. So this here is less than epsilon. And you have this inequality for all x. So you have this one here less than this for all x. And that's exactly uniform convergence. Yeah?

AUDIENCE: Where did we get the first-- when we said that the sum of these Mk's are less than epsilon, where did that come from?

TOBIAS COLDING: So let me just-- I'll get to this in just a second. But let me just say what is the strategy here. So we have this sequence. Maybe I erased the statement, unfortunately. So we have this sequence of functions on an interval. And you had that this function here was bounded like this. And then you had that these are just real numbers. And you had that these real numbers here were summable. So these are non-negative, real numbers, of course.

And so then-- and I claim-- so the claim is that the sum here of these guys, these here, I claim that these guys here converge uniformly to some function S of x. The first step is to say that they actually-- that there exists some function S, and they converge to that function pointwise.

In order to prove that, I proved that these sums here, for each fixed x, is a Cauchy sequence. Once there is a Cauchy sequence for each fixed x, then they will be convergent. And the limits are now defined to be S of x. So that gives me a function.

And then the thing is that I have to upgrade this convergence, which initially is pointwise to uniform convergence. And so now you are asking-- so now--

AUDIENCE: Why did-- so we got the sum of Mk's But then where did we-- how did we just [INAUDIBLE]?

TOBIAS COLDING: That's right. And so now-- so this here was just very general. And it just used this inequality. That's all it used. Now I'm using that these guys are summable. So this means that if I'm starting this summation sufficiently far out, then I can make it less than a given epsilon.

AUDIENCE: Oh, OK. Because it's technically--

TOBIAS COLDING: Because it's just like that. These are non-negative number. And that they are summable just means that if I go out really, really far, then from there on out, it makes very little contribution to the total sum. OK.

OK. So right. So right. So you see the first thing is, for each fixed x, I prove that these Sn of x is a Cauchy sequence that then allow me to define a function S of x. And I get that the sequence Sn of x is converging. The sequence function is converging pointwise. And then again, once I have that, then I need to upgrade it to be uniform convergence. And that's what we did here. OK?

OK. So now let's try to apply this. Yeah?

AUDIENCE: So what if the Mk's are kind of converging sequence but in the small side, the 0 side, is on a low index? So then, as I go further out, it's getting bigger.

TOBIAS COLDING: Can we look at an example and see if it becomes clear? So let's look at a sort of-- let's look at an example. Let's look at a very-- yeah, let's look at an example. So in fact, let's look at the function, the example we just looked at, where we said that it followed from Weierstrass M-test.

So what was the example I just looked at? So I take a function fn of x. This is now x to the n divided by n factorial. Now I define Sn of x. So this is now the sum from k equal to 0 to n of this fn of x-- fk of x, sorry. So this means that this thing here is the sum from k equal to 0 to n of xk over k factorial. OK?

And I want to show that this-- so I want to show that this Sn of x converges to the exponential function uniformly on an interval of this form. So L is some fixed number, and I'm looking between minus L to L. And I claim that they converge.

Now, let's try to apply Weierstrass M-test. And so what are the Mn? The Mn, in this case, the natural one is L to the power n times n factorial. If I'm looking at those numbers, these are non-negative numbers. And we already know that this sequence of non-negative numbers is summable, that these non-negative numbers, like this, that they are summable.

And suppose now I have-- suppose now I take this fn of x. So this is this function x to the power n over n factorial. x here, remember, is in this bounded interval. Well, let's look at fn of x. So this is xn over n factorial. But this thing here-- so this is just x to the power n n factorial. This is why it's lying in here.

So the absolute value is bounded by L. So this is L n over n factorial. So I have that this sequence is less than this Mn. And so now, Weierstrass. But we restrict to this interval here.

And so now Weierstrass M-test tells you that the sum here, when you're summing from k to 0 to n xk over k factorial, that these actually converge to the limit uniformly on-- But remember, we are restricting. And the limit here, we already have defined the limit to be the exponential function. Is that clear or not? Does that answer, or not really?

AUDIENCE: I'm not sure.

TOBIAS COLDING: OK.

AUDIENCE: So I guess my question was, is it that Mn must be a convergent series?

TOBIAS COLDING: Yeah. So Mn-- that's right. So Mn. Mn, in this example or in the general example, it's-- so why is the Mn so much easier to determine? Because they are just numbers. They are just non-negative numbers. And you're just asking if the series of non-negative numbers-- it's not function, it's non-negative numbers-- if that's convergent or not.

And so since they're non-negative, it just means that the sum is finite. And so that's much easier to determine. OK. So now we can do the same argument more generally. So this leads us to the following, that suppose you take a power series.

So suppose you take a power series. So you're summing here n equal to 0 to infinity a n xn. This is a power series So power series. Remember that a power series is where you're taking a sequence of numbers, real numbers, and you think about these as coefficients in this polynomial. And then you're summing over them. But it's like-- a n is the coefficient to x to the power n.

So this is a power series. And a power series again means-- so you think about it as a series because for each x, it's like a regular series. But then it's like the value that you get depends on x. And so for this, we defined the radius of convergence.

And let me remind you how this radius of convergence was defined. You take the a n, and you take the absolute value of it, and then you take the n-th root of that. And then you're looking at limsup as n goes to infinity. That gives you some number. That number potentially could be infinity. It's always non-negative, and it will always give you a number. It could be 0, it could be infinity, and it could be anything in between.

And then the radius of convergence-- of convergence of the power series is-- so this is right by R. And this is 1 over M. And where this convention in this case-- usually, you don't divide by 0 or infinity. But in this case, if M here is 0, this means that the radius of convergence is infinite. If M is infinite, then it means that the radius of convergence is 0.

And now the slightly surprising fact. But we proved this. But I'm just reminding you about it, that the thing is that the power series is convergent. So here you have 0. It's convergent inside. If this here is R, it's convergent inside here, and it's divergent outside. But it's not completely clear what happened on the boundary. So you have to check the boundary separately. And that depends on the power. There's no general formula for that.

So now the theorem here is that-- so the theorem is like this. And this is just generalizing this example that we just did. And the theorem is the following, that if you take a power series-- so if you have a power series and R is the radius of convergence and L here, it has to be strictly less than the radius of convergence, then these functions here, k equal to 0 to n a n-- ak sorry, xk, that these functions here converge to the limit here uniformly on this interval here, from minus L to L.

Remember that L had to be strictly less than the radius of convergence. And so this is extremely useful. If you have-- and it will be clear next time why it's useful because if you have uniform convergence, then it becomes very easy-- if you have uniform convergence, then you can argue about if you want to take the derivative of integrating this thing over here, if you have uniform convergence, then it will say that then a lot of operations will be more precise about that next time.

But a lot of operations, like integration and differentiation, you can do that over here and then go to the limit. So instead of if you were to integrate this thing here. So just say that-- but this seems maybe very, very hard. But the thing is, if you can integrate here, then you can just use the usual sum, the usual formula for the sum. It's just the sum of the integral. And likewise for differentiation.

And so, OK, maybe I will just-- yeah. Maybe I'll just stop here. I wrote out the proof in the notes. And I already posted the notes, the lecture notes. Are there any questions? OK. Great.