[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: So now let's talk about first the different concepts that we have talked about in class. And then in the end, I'll get to just some example of some problems that you might face on the midterm.

So the first is this-- that if you take a subset of Rn, to say that A is bounded from above, it just means that there exists M such that all a in A, a is less than or equal to M. So this is what it means to be bounded from above from a subset.

And now the least upper bound. And this is a number-- so M is the least-- so M is a least upper bound if M is an upper bound and any other upper bound-- let's call it M bar-- is bigger or equal to M. So this is what it means to be a least upper bound.

And so the least upper bound-- so the least upper bound-- this is what you denote by sup of A. This here is the least upper bound, and similarly For lower bounds.

And the real numbers have this property that if a set is bounded from above, then there is a least upper bound, and likewise for the set that is bounded below. So that was sup. And we'll get to this with the lim sup and lim inf. I'll just review that a little later also. But that's another [? long form ?] [? chain. ?]

A sequence-- so another key concept in this class is sequence. And let's begin with a sequence of real numbers. So this is a map from the positive integers into R.

And again, we denote-- so the image-- so if you're looking at the image of some n, we typically denote this or by a n or xn or something like that where we use the subscript, but it's just the image under this map So that's a sequence. So for each positive integer, you get a real number.

Then for a sequence, one of the first things you were interested in is convergence. So if this here is a sequence, that xn converges to some x here, this here just means that for all epsilon greater than 0, there exists-- I will use this symbol. I haven't used it until now, but it's just so hard not to use it.

This here means "there exists." It's not on the midterm. But it's a totally standard symbol. So there exists capital N such that if little n is bigger or equal to capital N, then xn minus x is smaller than epsilon.

So this is what it means for a sequence to converge to x-- sequence xn converges to x. And you write it like this, or you can write xn and then the limit here as n goes to infinity is equal to-- both of these are standard use. And then also, if a sequence is not convergent, then we say it is divergent. So a sequence is either convergent, or it's divergent.

And then there's this property-- useful property, so a lemma. If xn is convergent, then the set-- this here is a set-- this here is a bounded set. So we'll talk about that.

And then another thing about sequences is the algebraic properties of sequences. And what are the algebraic properties? They are that if you take two sequences a n and bn, and let's say that a n converges to a and bn converges to b. Then the algebraic-- and c is just a constant.

Then you have that-- so-- and often, one combines-- what I did as 1 and 2 before-- you combine, and you're looking at a n plus c times bn. This here converges to a plus cb. This is the first in this-- and when I state it like this, it's combining two of the 1 properties that we talked about earlier.

The second one is that a n times bn will converge to a times b. Third property is that if these here are not equal to 0 and the limit here is not equal to 0, then 1 over a n converges to 1/a. And so of course, this last one, [INAUDIBLE]-- this one implies, in particular, of course, that also, bn over a n converges to b/a. This is just because this property here combined with the property 2-- that will give you this.

So that's the algebraic properties of limits. Then you have the concept of a subsequence, so a subsequence. So we imagine that we have a sequence. And often we don't go all the way back like this, but a sequence, again, is a map from the positive integers into R. And we denote the image of n by, typically, xn.

Then a subsequence of this sequence-- and this is typically denoted something like this, where you have a further subscript. And this is important, again, that this is equal to f of g of k, where g here-- and this is the really important property-- that g is a map from the positive integers into the positive integers. And then, really important is that g here is strictly increasing.

And so strictly increasing guarantees you that these numbers here are taken in the same order as in the original sequence. But you may skip a bunch. You take one, and then you may skip a bunch, and then you take another one, and then you may skip more. But it's in the same order, and that's what strictly increasing is guaranteeing.

And then you have this property that-- so you have the theorem that a sequence xn is convergent if and only if all subsequences are convergent.

And one of the two directions for this theorem was trivial, because if you take a sequence, then it is-- if you're thinking about this map here g as being the identity, so that would give you a subsequence, but that really is the original sequence. So the original sequence is a subsequence of itself. So if all subsequences are convergent, well, then, in particular, the sequence itself is convergent. So one of the two directions was trivial.

Now, then we had these two theorems, and they were [INAUDIBLE]. So the thing in mathematics is that things build on each other. And hopefully, each step is fairly trivial, but you build step by step, and then you end up with something highly non-trivial. So this here was pretty trivial, but then we used it later on to prove something non-trivial.

So this was this monotone convergence theorem. And there were two versions of this. You had a sequence xn, and xn was increasing. So that means that the next element is bigger or equal to the previous element. If this here was the case, then xn is convergent if and only if xn is bounded-- if the sequence is bounded. And the sup of the xn-- this here is the limit.

And the second version of this is if you have a monotone-- so instead of being increasing, it's now decreasing. So the next element is smaller or equal to the previous element. And then similarly, the sequence xn is convergent if and only if xn is bounded from below.

And in this case-- and if it is bounded from below, then the limit here, [INAUDIBLE] of xn is the inf. So that was the monotone convergence theorem. And it has these two versions, the increasing version and the decreasing version.

Then another important concept was that of the Cauchy sequence. And the reason why Cauchy sequence is useful even for the real numbers-- later on, we generalize these concepts-- the most obvious [INAUDIBLE] of the concept of convergence, but also of that of being a Cauchy sequence-- to metric spaces.

But even for the real numbers, Cauchy sequence-- you might be interested in whether a sequence is convergent, but sometimes it's much easier to prove that it's a Cauchy sequence than to prove that it actually is convergent. And so if you don't really care about what the limit is, then it's often easier to prove that it's a Cauchy sequence.

So Cauchy sequence-- so you have a sequence. Again, we're still in the real numbers. And this is a Cauchy sequence. It just means that for all-- and now I will use also this here. This symbol just means "for all," and that's also a standard symbol. It's an inverted A.

For all epsilon greater than 0, there exists-- again, this here-- the inverted E means "there exists"-- delta-- I'm sorry. There exists capital N such that if little n and little m are bigger than capital N, then those two elements are bunched epsilon close together [INAUDIBLE]. This is what it means for a sequence to be Cauchy sequence-- that the tail of the sequence lies very, very close together.

Then let me skip. In my notes, I have something about contracting, but actually, there won't be anything about contracting methods, so I'll skip that. Let me now next talk about the Cauchy convergence theorem and the Bolzano-Weierstrass theorem.

So Cauchy-- also let me just remind you that, again, on the midterm-- so there are these five main problems. Then there's the six bonus problems. But make sure that you-- it doesn't matter which one you do, so you might as well start with the ones that seems easiest to you. And you can see, anyway, it says exactly how many points each of them get, so you can-- if you don't have enough time to finish all of it, you can be sure to try to optimize.

So what is the Cauchy convergence here? So again, we are in Rn-- [? R, ?] sorry. And we have n. So the Cauchy convergence theorem [INAUDIBLE] theorem. Let's say that if xn is a Cauchy sequence, then xn is convergent. That's the Cauchy convergence theorem.

The Bolzano-Weierstrass theorem-- and this Cauchy convergence-- we talked about that. So the concepts-- when we in a minute review metric spaces, then convergent sequence-- that makes sense in a metric space. Cauchy sequence makes sense in a metric space. But it's not-- and the statement of this theorem makes sense in a metric space, but it's not always that it's true. It holds in R, but it doesn't hold in a general metric space.

The Bolzano-Weierstrass theorem-- statement of that also makes sense in a general metric space, but it doesn't hold in a general metric space. So these are-- again, these are things about the real numbers.

They extend to more than the real numbers, but you have to-- for other spaces, you have to prove it individually. So Bolzano-Weierstrass theorem says that if xn is a bounded sequence, then xn has a convergent subsequence. So that's the Bolzano-Weierstrass theorem.

The next thing I want to review is what it means for a function to be continuous. So continuity is another important property. It's closely tied together with [INAUDIBLE] sequence, and we'll get to that in just a minute.

So if you have a function-- let's say on R, but it could be defined on a subset of R. So if you have a function like this, then we say that f here is continuous at x0 if for all-- it's continuous at x0 if for all epsilon greater than 0, there exists a delta greater than 0 such that, if x minus x0 is smaller than delta, then the images are smaller than epsilon.

So that's what it means for a function to be continuous. And there was this theorem or lemma that tied things together. So lemma-- if f is continuous at x0 and xn converges to x0, then the images of this sequences-- so that's another sequence-- converge to f of x0. So we proved this even in a metric space, but remember that we're still in R.

And then we had the algebraic properties of functions, and they were very similar to this. So let me just review them-- so algebraic properties of continuous functions.

So imagine that you have f and g are continuous functions. [INAUDIBLE] everything is defined on R. c here is a constant. Well, then, property 1 that corresponds to that property 1 is that this function plus c times this function here-- this here is also continuous. And then 2 is that if this function here is never equal to 0, then 1 over this function here is also continuous.

And then-- sorry, that was not the-- sorry, I should make that the third property-- sorry-- that f times g is also continuous. And then the third property would be that if f is not equal to 0, [INAUDIBLE]. Then this function here is also continuous.

And again, I divided it up. When I did it, I divided it up into four steps because I separated this out as two steps. And so from this one, we got that by-- just because then we proved that the constant function was continuous and the linear function, the function f of x equal to x, is continuous.

Once we had that, then we just use 1 and 2 here to prove that all polynomials continuous. And then if we bring in 3, then we have this with all rational functions are continuous except, of course, where the denominator vanishes [INAUDIBLE]. So that was how you built continuous functions from non-continuous functions.

Then we proved the extreme value and the intermediate value theorem. We had these two theorems. And they were proved-- so one way to prove it that we did last time was to use this lemma that tied together continuity and sequences.

And so we had the extreme value theorem. And this just says that if you have a function-- but now it's crucial that this here is a bounded interval and the endpoints are included in this.

So if you have a function like this and the function is continuous, then the sup and the inf over the set here-- this achieves a max or, if you would, max of f and min of f are achieved. In particular, the function is bounded. So I'll just state it like this, a little bit brief like that. So that was the extreme value theorem.

The intermediate value theorem-- it said that if you have a function-- imagine it starting off here and it ending there. So this is like a function. The intermediate value theorem says that everything in between these two is also the image of something.

So if-- let's say that f of a is smaller than f of b. Then f y here is between f of a and f of b, then there exists x between a and b so that f of x is equal to y. Again, the functions we talk about are continuous, and they are defined on this [? closed ?] interval-- this interval.

So that was those two theorems. And they used some property about that-- they used two things. They used the Bolzano-Weierstrass theorem, and then they used how sequences and continuous functions were tied together.

The Bolzano-Weierstrass theorem is using something about that the real numbers are ordered and like this with sup and inf [? R. ?] You have the least upper bound property and the greatest lower bound property.

Now another thing-- so we talked about-- so some of the highlights in terms of concepts are sequences and continuity of functions, and then various properties that follow from this.

Then another thing is series. So that's another key concept. A series is where you're starting with a sequence. So series-- so you're starting with-- so this here is a sequence.

And now you form another sequence Sn. But here you're summing, and it could start wherever you want to start it. Let's say it starts from 0. You're summing the first-- in this case, it would be-- because you're starting at 0, you're summing the first n plus 1. So a i-- this is a0 plus a n-- sorry, n plus 1 in this case. But you don't have to start at 0. If you started at 1, it would be the first n numbers.

So you're looking at-- so a series is really kind of the same thing, but you think about it a little bit different, but it's really the same thing as a sequence, where you're starting with a sequence and then you form a new sequence.

And then the key question here is-- so key question-- is a series convergent or not? And so that's a key question. And if it is convergent-- and you often-- the series is often written like this.

And so this here will symbolize the series. And if the series is convergent, it will also symbolize the limit, so it has two meanings. But even if it's not convergent, then it [? worked. ?] Well, if it's not convergent, you would just write down-- and you would think about this as a series.

Now, there are some series that are of particular importance, and they can then be used to determine whether or not other series are convergent. So let me just review them.

And the most important of all is the geometric series. And the geometric series is the series c to the n, n equal to 0 to infinity, and your c is just some fixed number, real number, and you're looking at this.

And this is the geometric series, and this is convergent if the absolute value of c is less than 1 and divergent otherwise. And if it is convergent when c in absolute value is less than 1, then this sum here is equal to 1 minus c.

But remember two things. One is that when you actually want to evaluate the sum, it's crucial where you're starting. If I started at 1, there is not any more equal to this. But you have to subtract the first element that was missing. So it's really-- for convergence, it doesn't matter where you start, but for the limits, it's key.

That's one thing. And the other thing is that when you-- again, we talked about this-- that when you're given a series, even before there's any question, the first thing-- when you have a series, the first thing you look at is, does this a n converge to 0 or not? If a n does not go to 0, well, the series is not going to converge. There's no need to apply any fancy test. And for a series, the question is always whether it's convergent or not.

So that's a geometric series. And then there are two other series that we talked about. And let me just-- not only talk about, but we also determined whether or not they were convergent, and then we used them to determine whether other series were convergent. But again, the most important of all of them is the geometric series. That's the one that we used to determine whether power series in general was convergent.

But there are two others that are worth mentioning. One is the harmonic series. The harmonic series is the series that is summing 1/n. So this here is divergent. So this follows second.

And then there is a closely related-- let's call it 2 and 1/2. That's called the alternating harmonic series. So the alternating was that it was like the harmonic series, but the sign of it was alternating. And this here is actually convergent. And then-- so there's 1, 2, 2 and 1/2, and then there's 3, and that's this series here. We looked at this series here, and this series was convergent. The only one where we determine what the sum was, was actually the geometric series.

So that's some key series. Then there is-- when you have a series, there was one concept that proved to be quite useful, and that was the concept of absolute convergence [INAUDIBLE].

And so you were looking at a series. And absolute convergence was that this sum here was convergent. So this here is absolute convergent if this series, where you just-- the a n is replaced by the absolute value-- is convergent.

So of course, for a series of non-negative numbers, it's automatically-- if it's convergent, it's automatically absolute convergent. And we talked about it a little bit that one of the things about an absolute convergent is that you could reorder the way you're summing things, and it wouldn't matter for the limit. But if it's not absolute convergence that really matters. We talked about it. We didn't really prove anything there, but I think it was clear that you could do this.

One thing where we used it was when we looked at the product of series. Then the product of series was convergent if each of them was absolute convergence. And there you saw that we estimated some remainder term. And there, it was really key that we had absolute convergence.

So anyway, so absolute convergence is an important-- for series, it's a stronger property than convergence, so absolute convergence implies convergence. Maybe I should write this as a theorem. So we had the theorem-- so absolute convergence. So this implies convergent, but it's stronger. In general, it's stronger.

And now-- then we had some tests. Then we had some tests. And so we talked in particular about-- there were a few other [? tests ?] that we talked about, but I didn't-- they're not actually going to be on the midterm. And some of them, we'll get back to. This one here won't be on the midterm [INAUDIBLE] alternating. But we talked about how to improve the convergence.

And the important-- and when we talked about it, then we discovered that this thing here-- that there was a stronger statement that if you have alternating and the a n are all non-negative and these guys, it's decreasing-- goes decreasing down to 0-- then actually, any alternating sequence like that would be convergent.

Anyway-- but for the test we really talked about was two tests. One of them had two parts. And then, in a minute, we'll talk about how we apply them to power series, [INAUDIBLE].

So we had some tests to determine whether or not a series-- or not-- a series is convergent. So what are the tests? So there are the comparison tests.

Let me just first write down the test. So the comparison test-- and there was two versions, version A and version B. Then there was the ratio test, and then there was the root test. The ratio test and the root test-- they are both offsprings of the geometric series.

In both of these, you got-- let's just talk about these first. The ratio test was when we were looking at this-- the ratio of a n plus 1 over an. So you are implicitly assuming that the a n does not vanish. I'm not sure. So you can form this ratio.

And then if this limit here as n goes to infinity-- if this limit here is less than 1, then the ratio test says that the series was convergent. If the ratio was bigger than 1, then this ordinary element a n would not even go to 0. So this here is-- so in this case here-- maybe I'll write it over here.

So for the ratio test-- so the ratio test-- you assume that the a n-- none of the a n equals 0. And you're looking at a n plus 1 over a n. And if this limit here-- let's call this limit-- let's assume that the limit exists. Of course, it may not exist

But if the limit existed-- so if c here is less than 1-- it's always non-negative because these are non-negative numbers-- if c is less than 1, then the series is convergent. If this c here is strictly bigger than 1, this implies that the series is divergent. And if the c was equal to 1, then it's unclear-- and so unclear, which means that further tests are needed.

And so now-- so that's the ratio test. And the root test was similar. The root test was-- so you're looking at the a n. You take the-- so this is the root test. And again, this is tested by comparing it to the geometric series-- so the root test. So you take the a n, take the n-th root of this, and you're looking at the limit here. And we actually looked at the lim sup. I'm writing the lim sup. The reason why I want the lim sup is that I want to make sure that it's-- later on, I want to apply more generally.

So if the lim sup here was less than 1-- let's call the lim sup-- let's call it c. If c here is less than 1, then the series is convergent. 2-- if the c is bigger than 1, series is divergent. This fails the most basic test-- divergent. And again, 3-- if c is equal to 1, then it's unclear. Further tests are needed. That was those two tests.

The comparison tests-- let's get back to that. The comparison test has two parts that you had. Suppose you had a n and bn. The first part is this statement. If you have two series, one based on the a n, another series based on the bn-- if you have this-- if bn-- so this here-- if this series here is convergent, then [INAUDIBLE] is convergent.

And this is sometimes used both exactly like this, and it's also sometimes used where you know that this one here is divergent, and so you compute that this one is [? divergent. ?] The other part of the comparison test was if you had the limit here, if this limit here exists and it was something L, then actually, it [? says ?] that-- let's say L here is not equal to 0.

Then if one of them is convergent, the other one is also convergent. So if a n-- or bn, let's say-- bn is convergent, then it implies that a n is convergent. And vice versa-- this actually-- [INAUDIBLE] this is convergent [INAUDIBLE].

So those were the three tests that we really went through. There were a few other tests, like, again, this alternating series test that we saw a proof, but it's not going to be on midterms, and we will return to this later. Yeah?

AUDIENCE: Are there patterns of when we should apply which tests?

TOBIAS COLDING: Yeah-- no. And I want to get to this in just a second. Let me just say a couple of words first, and then let me jump to some of the examples of problems, and then I'll come back, and then we'll do a little bit more. But let's just talk about power series, and maybe-- let me think [INAUDIBLE] and whether-- just now when you were asking the question-- no, let's just review the power series.

So our power series was a series, but it was something that also defined a function. And so you take-- so you have-- so a power change. So you have a sequence a n. And then you think for each x-- for x, think about x as being fixed-- you form a series, and this is the series that you form. And so for each x you can ask, is this series convergent or not?

This here is called a power series. And the key point for power series is, again, of course, whether or not this is convergent. And this gave rise to the radius of convergence. And so the radius of convergence was-- that we defined-- we looked at lim sup of the n-th root of a n, like that.

And we looked at when this thing here-- if this thing here is-- actually, we defined this thing here, and-- so we got this thing here. And then we defined the radius of convergence. So the radius of convergence-- this is R, and the R is 1 over this lim sup.

But there was an understanding in this. The understanding is that if this number here-- this number potentially could be infinite. If it was infinite, it meant that this thing here was 0. If this one here is infinity-- if this here is 0, then this here would be infinity. So that's like a convention in this particular case. Usually, we don't want to divide by 0 [INAUDIBLE]. But this is a convention in this particular case.

And so using this radius of convergence, we saw that if you take a power series, if you have any power series like this-- here you have 0-- any power series like this, it's going to be-- if this here is R-- and again, R could be 0 or it could be infinity.

But if this is R-- sorry, I should write this with a minus sign. Then for x in between-- for x not including the endpoints, but in between-- the series is convergent. The power series is convergent. And outside here, it's divergent. Outside is divergent. Here it's convergent. And at those two points here, unclear. [INAUDIBLE].

So this is always this picture. So this comes back to your question about which tests-- of course, you can use any test you want, but when I-- like if you're looking at a series, you can use whatever test you want. But if it was a power series, you already know that there's some interval around 0 where it's convergent, and then there's a question about the endpoints.

And so now let's look at some examples of this. And suppose you look at-- and so It's usually not too hard to use either-- this here was based on the root test, so you use the root test or the ratio test for power [? convergence. ?] But you can take advantage of it sometimes it's just slightly easier.

So let's look at two examples here. So suppose that you're asked to find the radius of convergence for the power series. And then let's say that this is n to the fifth power, xn. So suppose that you are-- this is a power series you're given, and you want to find the radius of convergence.

You can either just plug it into this test, or you can just say, OK, we'll just apply it. We know that it's some interval, and I just want to see for which x does this here converge. So if I looked-- in this case, you might find it slightly easier to do the root test.

So if you're looking at the n plus 1 element in this series-- n elements would be this to the fifth power. Think about x as being fixed, and you're asking for which x. And then you want to-- for a given x, you want to determine whether the series-- this series here-- is convergent.

So if you were using the root test, you would look at this thing here. And you might find in this case slightly easier to use the root test. So if you're looking at this, this thing here is now n plus 1 to the fifth power times x to the n plus 1 divided by n to the fifth power times x to the n. That's that ratio.

And so of course, you can write this here. This is like-- sorry, n plus 1 multiplied by itself five times, divided by n multiplied by itself five times. And then this thing here-- you can write it as x to the n, times x, over x to the n.

So you see, this thing here, you can then think about as n plus 1 over n to the fifth power, and times x. And I can write this thing here in the usual way. If you want, you can write it as 1 plus 1/n. I'm dividing by n in both denominator. And denominator, you can write it like this-- to the fifth power times x.

And so now-- the root test-- it's asking whether this-- the absolute value of this root test would be then asking if the limit here of x n plus 1 over-- I'm sorry, a n plus 1 over a n-- if this here has a limit. And then it's asking you, if that limit is less than 1, if it has a limit and the limit is less than 1, then the series is convergent, and it's divergent if it's bigger than 1.

But in this case, this thing here, this ratio here, is 1 plus 1/n to the fifth power, times x. But this is all non-negative, so the absolute value is just going over here. The algebraic property of the limit says that this one here goes to 0, and so raised to the fifth power still goes to 0. It's multiplying a bunch of sequences five times together. That gives you 0. To 1, [? too-- ?] so this here this goes to 1, and so even raised to the fifth power, still goes to 1. So this here goes to the norm of x.

And so the ratio test would be that if x here-- so the ratio test gives you-- is that if this norm here is less than 1, then this series is convergent. If the norm here is bigger than 1, the ratio is divergent. And it's unclear if the norm is equal to 1.

So in other words, you see that the ratio test here gives you directly that the radius of convergence is 1. And of course, you could apply the root test. The root test is great on this here. It's a little bit more-- no, you have to do a little bit more argument. It's not very hard. But it actually-- in this case, the ratio test would be easier to-- it would be less to argue than in the root test.

Let's do one more example of this. So maybe I'll just switch a little bit [? to-- ?] and then I'll come back and review some more stuff. I'll just talk about a few of the problems that I added to the end of my review notes. Again, I'll post this with these problems. And obviously, they are-- it's not exactly problems on the midterm, but [INAUDIBLE] it's certainly relevant to the midterm.

And also, another thing I want to say is that the homework is graded up to and including 4. So I wasn't actually aware of, over the weekend, they were supposed to be handed back within, latest, a week of when you handed it in, and I just discovered it over the weekend that it was-- some of them were late, so-- but it's fixed. I'm not the grader. There's two [INAUDIBLE] that are grading and a grader. But it's all fixed now. So the first four are-- you can review those also for the midterm.

So the other power series I want to look at is-- so again, what is the radius of convergence? The nice thing about power series is that power series also test about series. [INAUDIBLE].

So what is the radius of convergence for-- and now the series is a slightly different series. It's this series here-- so this series. And so now let me-- again, you could apply the root test.

The root test is the one that just gives you an answer. It always gives you an answer, but you may have to work harder to find the n-th root, and sometimes it's just easier to use the ratio test. So let's use the ratio test again. So a n here is n to the fifth power, x to the n, over n factorial. So if you're looking at a n plus 1 over a n-- maybe I should-- maybe let me write it over here.

So if I'm looking at a n plus 1 over a n-- so then I have the n plus 1 to the fifth power, times x to the power n plus 1, over n plus 1 factorial. That is a n plus 1. And then I have to divide by this, but when I divide by a fraction, it's like multiplying by the inverse of this. So like this thing here-- [INAUDIBLE] the n power, x to the n power-- like that.

And so again, [INAUDIBLE] I can write this thing here as-- so if I just do like I did before, I would have that n plus 1 over n, to the fifth power. That's when I take this one here and that one together. Then I have-- when I take those two together, I can get rid of the n here, and I just get an x.

And then when I look at the ratio of these two, this one here-- n plus 1 factorial-- you can think about that as n plus 1 and then times n factorial. And so this n factorial cancel out with that, and you're left with an n plus 1. So this is what you get.

And this one here-- just writing it as we did before, dividing this fraction by 1/n in both numerator and denominator, getting this thing here to the power of 5 times x n plus 1.

And now you see that-- and this here, I guess it should have been absolute value [? applied, ?] so it's absolute value on this, like that. This one here converged to 1. This one is just x, so this converged to x. And then this here goes to infinity, so the ratio here goes to 0. I'm using the algebraic property of limits. So this is converging to 0.

And so you see that it does have a limit, so the ratio test [? applied ?] because that limit is 0 for all x, it's going to be convergent for all x. So the radius of convergence in this case-- and of course, on the midterm, don't just say that the radius of convergence is infinite. Explain this thing here, and the more detail, the better, because even if there's some typos, you definitely get should definitely get-- I'm not grading the midterm either, but you definitely should get partial credit. Yeah?

AUDIENCE: So to get R, is it always like you divide by the lim sup? Because-- is the thing on the bottom one the lim sup or just the lim?

TOBIAS COLDING: Yeah. So when you take just a complete general series, then the radius of convergence is by definition this. This is related to-- you asked this question a couple of times ago that-- because we started with the ratio test. We did that before we did the root test.

And you were noting that in the ratio test, you're always comparing an element to the previous one. And in general, the series can be convergent even though the next element is slightly larger than the previous. If that keeps happening, it's not convergent, but it could happen a few times, and then it goes really down, and that's fine.

And so that's why I like the root test-- gives you this general result about that it's always-- it always works. It doesn't assume that the a n's are not 0, et cetera. For any particular-- and it really should be the lim sup here. And then it's defined. It's a number that's always defined.

And again, the way it's stated here, the way the proof goes, is it's not comparing two consecutive a n. In any particular example, the root test may not be the easiest way to argue. Often, if you can, for a power series-- almost always, I would say, in a power series, it's almost always easier to use the ratio test, but it requires that the power series has some nice property for this to apply. [INAUDIBLE] Yeah?

AUDIENCE: So [INAUDIBLE] R is infinity for that example, it was because it was convergent for all x [INAUDIBLE].

TOBIAS COLDING: So in this example here-- then yeah, this here is infinity, which means that this thing here would be 0. Yeah?

AUDIENCE: But you know [INAUDIBLE].

TOBIAS COLDING: But I didn't actually calculate. You see, I didn't calculate the [INAUDIBLE] in this example. But if I had calculated, I would actually-- and it's not very hard to calculate, but it needs some small tricks, and so-- and we haven't-- yeah, we haven't talked about. So that's why I [INAUDIBLE].

Let me talk about a few other things, and then-- so again, I don't think I can [INAUDIBLE] review all what I had in my review notes. But let me just talk about a few of the problems that I had. But again-- so the things I want-- let me just quickly mention the things I may or may not get to that I have in my review notes. So I had also [INAUDIBLE] property-- what is lim sup? What is lim inf? And so definitely review that also. Talked about the radius of convergence.

Well, maybe this is all, and then talk a little bit about the power series for the exponential functions. But again, all the stuff we did on the power series for the exponential functions-- at this point, we have established this.

This is like-- it's more like-- there won't be anything particular about them. The exponential function itself could be on the midterm, but it's not like you would have to prove the properties of the exponential function. This property-- that it has the property that the exponential function of x plus y is the product of that kind of stuff-- it's not on the midterm. So I didn't-- so it's not part of the review.

The other thing that I probably won't get to, because in a minute I'll just turn to these other problems, is that metric spaces-- definitely review metric spaces. It was in the last lecture. So there's a bunch of-- there's three properties for a metric space. The distance should be non-negative, it should be symmetric, and the distance between two points should be 0 if and only if those points are the same. And then the crucial property in the metric space-- then that's like the powerful property-- is the triangle inequality.

And so there was a few examples. And review those examples. They are in my review notes. The most important example, I would say, is-- the French rail metric is kind of a crazy metric. That's not one of the important ones. So I just did review notes as just the most important ones. And then there's this with, what does it mean to be convergent in a metric space, and what does it mean to be a Cauchy sequence?

And so now let's get to a couple of the other problems that I suggested that you look at. And so here are two other problems. And again, I won't get to-- there's also problems about metric spaces, and you should definitely look at it, but I don't have the time now, even though it's relevant for the midterm. I just have to pick some. Again, all of the problems that I-- I have four problems here. All of these are relevant for the midterm.

So now-- suppose you take a sequence. So I have a sequence here, xn. And so often, a sequence is defined recursively. So recursive-- so recursively defined [INAUDIBLE] sequences. So what does it mean? It means that typically, you have-- the first one is defined to be whatever, and then the next one is defined from the previous one.

And so the one I was suggesting looking at here is that-- you see this here, the first one [INAUDIBLE]. And then-- actually, [INAUDIBLE]. And then the next one-- let me just do this one here. It's actually simpler than what I have in my notes, but maybe that one, but I just want to-- so the next one is defined to be xn over n.

And so then-- so now, the question would be, for what a does this have a limit-- the sequence have a limit. And so you see that in this case-- we only have four minutes left, so I want to [INAUDIBLE]. But this is also-- I just wanted you to review what it means for a sequence to be defined recursively.

And so you see that in this case, you have that this thing here-- it is going down. It's actually going down. It even could go down by a fraction. But now, it's like-- if you're looking at this in here, you're looking at xn, this here is definitely going to converge, whatever you start with.

So I claim that you will always get that xn here converges to 0. And it's just-- you see that if you-- when you have something recursively, it often can pay to just write out the first few terms. So you have [? x1, ?] a. x2 is equal to a. It's also a. But the previous one, [? x3 ?] is now a/2. x4 is then a/2, divided by 3, so this is a/6, and so on.

And so if you continue like this, you will see that they get-- that it's like a divided by some larger and larger number. But you have to, of course, write it nicely out as a proof. But this is proving that. But again, this here-- if you just did what I did, that would not be enough to get full credit, so definitely write out.

The other thing, in the last minutes here-- the other thing-- the other example here [INAUDIBLE], suppose you take a series like this, and suppose that a n here is less than 1 over n squared. And then the question could be, is the series-- suppose you knew this, but that's all you know. Is the series convergent or divergent?

And then I say in the notes here-- I mean, in the review here-- that either you should prove that for all such series, it's convergent, or if that's not the case, you should give an example of a series with this property that is divergent. Because obviously, if all the a n were 0, it would be convergent. But it still could be another series with a n's that have that property that was divergent.

But in this case, it is going to be convergent. And why is it convergent? Because this series here is convergent. We have already seen that. So this here would be like the simple comparison test, like the comparison test.

So these are just examples of things you should be reviewing. They are-- like these things here, the two things to the end, like this one here-- this is super fast, so don't expect it to be quite as fast on the midterm. But it just gives you an idea of what you should be doing.