[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: But let me just quickly review some what we have done so far in the class. So first, we talked about R as an ordered set and fields. Then we have talked about sequences and limits. And then we talked about series. And then, finally, we have talked about-- and this is where we will continue today-- we talked about metric spaces.

And so the idea here is that we want to-- there's a lot of things from R that actually generalize and make sense. And some of those are sequences and limits. So they generalize from R to metric spaces. There are other things that don't generalize. It's not an order. A general metric space is not-- it doesn't necessarily come with an ordering. And it's not like a field. And in particular, typically, you can't sum elements. So a series doesn't make sense either. But these things here make sense in a metric space.

And so let me just remind you what is a metric space. So a metric space is a set. So x is a set. And then there is a distance function. So this is a map from x cross x into r. And for each two pair of elements x and y, in x you have that-- let me make it a capital X, like this. For every two elements in this set, the distance between them should be positive and it should be nonnegative and equal to 0. The distance between two points should be 0 if and only if these points are the same. And then it should be symmetric. So property 2 is that it should be symmetric. So it doesn't matter whether you go from x to y or from y to x. The distance is the same.

And the third, which is the most nontrivial part, is the triangle inequality. So this here is symmetric. And the third property is that if you're looking at x, y, and z, so you have three points, and you go from x to C, then it's certainly not further than if you first go to from x to y and then go from y to z. So this should be less than or equal to the distance from x to y plus the distance from y to z. And so this third property here is what's called the triangle inequality. So this is the triangle inequality.

Now so that's what a metric space is. And again, so the canonical example of a metric space is where you're taking x equal to the real numbers. And then for two true real numbers, the distance between them is just where you're looking at the norm of the difference. So this here is a nice distance. It's clearly nonnegative. It's clearly equal to 0 if and only if x is equal to y. It's clearly symmetric. And the triangle inequality is this inequality that we have used all the time.

So this here is slightly more fancy version of the same is where you're looking at R2. So the x here is R2 Or it could even be a general n. And then if you think about two elements in Rn and you think about, say, x here as-- it's written as x1, x2, up to xn, where each of these xi's are in R, and there are n many of them, and likewise for y.

And then the distance here the natural distance is where you're taking-- this is what's called the Euclidean distance. It's given by Pythagoras. And so it's the square root of the sum of the squares, so x1 minus y1 squared plus x2 minus y2 squared plus and then all the way up to xn minus yn squared.

And so you see that if n here is equal to 1, well, then each element is just a single element in R. And so then this thing here is really the same as that because it's the square of x minus y. It's the square of this distance. And then you take the square root. So it's a norm. So both of these is called the Euclidean-- usually referred to as a Euclidean distance.

So we looked at a few other examples. And I won't look at all of them again, but just two more. One is this with the box distance. And let me state it only in R2, but just like this, you could state it in any Rn. So the spaces are true, and the distance between two elements-- so again, we write it like this. And we think about x as being x1 comma x2, where these are real numbers, y equal to y1 comma y2.

And then this distance here is the sum of these distances. So this is a plus x2 minus y2, like that. And this is clearly, again, symmetric. It's clearly nonnegative. It's clearly equal to 0 if and only if x and y are the same. And the triangle inequality-- well, that just follows quite easily from the triangle inequality for this distance.

Then there is one more example. I mean, there's other-- but there's one more example that is often useful to keep in mind. And this is the last one I will review. And so this is the continuous function. So if you take an interval a and b are real numbers, and you're looking at the interval from a to b where both endpoints are included, and then you look at the continuous function on this interval.

And then if you have two elements f and g-- so these are continuous. Each of them are continuous functions. Ther the distance between these two continuous functions is equal to the max over x And then f of x minus g of x, right then, so x here is in the interval from a to b.

And so this thing here-- so this here was on the midterm, this particular example. And this thing here that the max is achieved is just like if you're looking at these two functions, then this is the difference between the two functions. So this is also continuous. And this means that the maximum at the minimum by the extreme value theorem is achieved on an interval of this form. So this thing here is really achieved, and it's clearly nonnegative. And it's easy to see that it has the other property of a metric space. So these are just the examples that we talked about. And I just want to remind you.

And so then the question is what generalizes from-- so we started with a class talking about R. And so it was an order field. But a lot of the things we talked about for R, the subsequent things we talked about, sequences and limits, they, as we will see, make total sense in a general metric space and has important consequences, including to solve differential equations and solving all kinds of equations, including differential equations.

So now let me remind you also, so we talked about that if this is a metric space, then a sequence in a metric space. This is just a map. This is a map from the natural numbers into x. So a sequence is like a map like this. And so usually, the image of the n element, this we will typically denote, again, like this, xn. So this is what a sequence is.

A subsequence-- this is defined very similar to what we did on the real line. So a subsequence is the composition of f with a strictly increasing map g from the natural numbers into n. And then the sequence is then where you're looking where you apply first g, and then you apply the f. So when you first apply g, then it gives you another natural number. And so this thing here, the image of this is then a element in x. So that's a subsequence.

And again, this was strictly increasing guarantee you that the elements appear in the same order. You're picking a selection of them, but they appear in the same order as in the original sequence.

So sequences and subsequences make sense. We can then also talk about-- and I know we talked about, but I just want to-- because of the break, I just want to review quickly a few of the things we talked about right before the break. So one is that a convergent sequence in a metric space-- so if this here is a metric space and xn is a sequence in xd, then we say that we say that xn converges to x if the following hold.

That for all epsilon greater than 0, there exists capital N such that if little n is equal to capital N, then xn the distance between xn and this limit should be less than epsilon. So that's exact parallel to how we defined the limit of a sequence or that a sequence is converging if it was a sequence in R.

And likewise, we can define a sequence in a metric space. So we define a Cauchy sequence in a metric space x, d. So this is a sequence so that for all-- so this is a sequence xn in such that for all epsilon greater than 0, there exists a capital N such that if little m and little n are bigger or equal to capital N.

So remember that a Cauchy sequence means that the tail is bunching together. So this means that if you are sufficiently far out, then these elements, xn and xm, are epsilon close together. So this is what it means for a sequence to be a Cauchy sequence. And so now we have this theorem, and I think I mentioned it before the break. But let's just quickly run through it because it's very short.

So if you have a-- so if this is a metric space, then the theorem is that any convergent sequence is also a Cauchy sequence. And the proof is like this. So given epsilon, since xn is a convergent sequence, so you assume that x is a convergence sequence-- let's call the limit x. Since xn converges to x, this means that there exists capital N such that if little n is bigger or equal to capital N, then the distance between little x, little n, and nx here, this distance here is less than-- and then I use epsilon over 2.

And so this means that, therefore, if little m and little n are both bigger than capital N then the distance between the two-- distance between these two elements here, well, by the triangle inequality, we have that this distance is less than if you go over the point x, so if you go from xn to x and then from x to xm like that.

But each of these, because both of these, both little m and little n are bigger than capital N, then both of these distances are bounded by epsilon over 2. And so you see that this thing here is smaller than epsilon. So this proved that in a metric space, any convergent sequence is also a Cauchy sequence.

And so we know that if the metric space is R with the usual Euclidean distance, then actually, you also-- and that was the convergence theorem, so the Cauchy. convergence theorem. So Cauchy convergence theorem-- it said that if x comma d is R with the Euclidean distance-- so I denote the Euclidean distance like this-- then we have the converse. So i.e. that any Cauchy sequence-- this is also convergent.

Now we looked at a very simple example. And if I have time, I will talk about it a little bit more interesting example. Otherwise, I'll talk about that next time. So if the metric space was R with the Euclidean distance, but then we had a converse-- but if the metric space was, say-- so we looked at where the metric space was-- so where x here, the set was the interval from 0 to 1, and the distance here was just the usual Euclidean distance.

So then if you're looking at the sequence xn, 1 over n, this one here, this is a sequence in this space. It's a Cauchy sequence. But it doesn't converge in this space because 0 is not part of the space. So this here is a Cauchy sequence. So you may say this is a rather trivial example, not very interesting. I'll show you another example a little bit later or next time that is more interesting but, in some sense, has some of the same feature, that the space is just not quite large enough. But it will seem more interesting. So this here is a Cauchy sequence that is not convergent.

So again, the Cauchy [INAUDIBLE], so this leads to the following. This leads to the following notion here. So this is the following definition of being Cauchy complete. So a space or metric space x comma d is Cauchy complete if every Cauchy sequence is convergent.

So now let me define some-- so if we begin with extending some notions from our-- so the first one I want to extend is what it means in a metric space, what is a form? So we have here-- so a metric form. So we have a metric space. We have a point in the metric space. We have some r which is a real number, positive real number. And then we define a Br of x here. This here is the set y in x So that the distance from x to y is less than or strictly less than r.

And so this here is called this here is called the metric ball. This set here is called a metric ball or just a ball with radius r and set x. Now let me just illustrate it. So a lot of the intuitions about metric spaces-- but you have to be careful because, sometimes, that intuition fools you just because metric spaces are allowed to be very general. But a lot of the intuition about metric spaces come from just thinking about things in Euclidean space.

So when I'm talking about Euclidean space, I'm thinking mostly about it could be on the line. So in the case where x is r or it could be in a plane where x is r2, or it could be where x is r3. And so in either of those three cases, the distance is the distance that you're used to, which is the Euclidean distance we talked about.

And so in that case, if you take a point here-- so just thinking about the example where you think about the example where x is R2, then a ball here-- well, a usual ball is really just everything that is the distance r from x. So this is kind of what I defined over here.

There's another thing you could also look at some other sets. So imagine that you take a line in the plane, and you could look at-- or Maybe this could even be-- it would look more interesting if I'm thinking about a line in R3. So imagine that this is now-- this is supposed to look like in R3. And then you could look at a line, and you could look at-- instead of looking at a metric ball, you could look at a tubular neighborhood. So suppose you're looking at-- that's very similar to a ball.

So a tubular neighborhood, let's say a r tubular neighborhood-- this is where have some positive number r and you're looking at everything that has distance at most or let's say strictly less than r from this line. So if you do that, that would look like a cylinder, where the boundary of the cylinder is not included because I require things to have distance strictly less than r. This here would be a r tubular neighborhood of a line.

So this is just to say that of notion here of a ball, you could even have something more general. And both of these things that you know well from Euclidean space, they make sense in just a general metric space. The line maybe doesn't make sense. Sometimes, often, it does, actually. But that's something more complicated. But you could just think about it as a subset. And then things that have distance strictly less than r to that subset.

So this is what it means for-- this is what a ball is, metric ball. Then there's the notion of a subset of a metric space being bounded, so a bounded subset. So there's another definition-- bounded subset of a metric space. So we have a metric space here. And we have A as a subset. And we say that A is bounded if there exists x, an element in this metric space, and d-- d is some positive real number such that A here is contained in the ball of radius d centered at x.

Now with this-- so there's a few things like this that we'll see for these things. The empty set-- if you take x and you take the subset that has no elements, that's the empty set. The empty set is contained in everything. So actually, the empty set would be bounded. But it's not a particularly interesting example of a bounded set.

And it may be that the whole metric space is possible that the whole metric space is bounded if it was like an interval. If it was just an interval where both-- so suppose you have some interval from A to B. Then this thing here is-- if you think about the usual distance on that interval, then that's a bounded set. The whole set x is bounded. So in particular, any subset would also be bounded.

So now let me just prove one theorem, one lemma, that is another thing that generalizes from R. And that is that if you take a Cauchy sequence-- so any Cauchy sequence is bounded. So let's prove this. So we have, again, a metric space. xn is a Cauchy sequence. And then we think about the set xn minus-- so this is this subset of x. And I claim that this subset here, this subset here-- so A here is bounded.

Now why is that the case? So since xn is a sequence, if I choose epsilon with-- so with epsilon, say, equal to 1, you have that there exists capital N so that if little n-- such that if little n and little m bigger than capital N, then the distance between the two is smaller than 1. This is just using that as a Cauchy sequence and just using one particular epsilon, namely taking epsilon to be equal to 1.

So now, of course, this means-- and now you don't need this event for all of this. You don't need it like this. So in particular, therefore, as long as n is bigger than capital N, well, you can use one-- or you can use m here. You can use capital N for m. So then this here-- say, therefore, for little n bigger than this, we have that this thing here, the distance to this is smaller than 1.

And so now you have that if you now set r equal to 1 plus the maximum of the distance from xN, x capital N, to xi, where i here runs from 1 of 2 capital N minus 1. So there are n minus 1 of these that you take the maximum over.

So if you define little r to be like this, then obviously, little r is some positive number. It's finite. And now I claim that our set A here is contained in this ball here around xn. So I claim this thing here.

Now let's see. So why is this the case? So if x here-- so we need to prove that-- we need to show that for all n you have that the distance here is less than r. If that's the case, then the whole set here would lie in this ball. But you see that if n here-- well, if n is bigger or equal to capital N, but this-- so we need to show this in here. If little n is bigger than capital N, then we have that the distance from this little n to this one here by the tail-- the tail was less than 1.

But this thing here is actually why-- this is actually-- this is equal to r because it was 1 plus this maximum. And then this is maximum over nonnegative numbers. So you see that for the whole tail, we have this property. But for all the other ones, for n less than or equal less than n minus 1, these are the remaining ones. There, the distance from xn to x capital N, where it's-- this is one of these guys. So it's clearly this one here. It's clearly less than capital R. And it's really strictly less than because there's this one also. Yeah?

AUDIENCE: [INAUDIBLE] does it has to be 1 plus the max, or could it be the max of 1 or all of those elements of the sequence?

PROFESSOR: I wanted this one here to be strict. It's just a silly thing. But that's why I took it like that. Because otherwise this would be strict. But it's just a trivial thing, and it doesn't matter very much. At this stage, it makes no difference, actually. But it was just to use this. We'll talk a little bit more about the balls because there is another way you could define the ball. Maybe I'll say a few words about that now.

So what this proves, again, is that in any metric space, a Cauchy sequence is automatically bounded. And so remember, in any metric space, a convergent sequence is a Cauchy sequence. So a convergent sequence is also bounded. And now there's this question, which is a natural question, and this is connected to the definition of ball.

So the way we define-- and it's actually worthwhile emphasizing because it's useful in the next thing that we want to get to. And so when I define a ball, I assumed-- so the definition of the ball-- just write it up again here. So the definition of a ball in a metric space-- these are the y in this set so that the distance to the center is strictly less than r.

You could, of course, have defined it differently. You could define it that this is strictly less or is there strictly less than or equal to-- I want to use strictly less here. But sometimes, if I want to be very clear, I will talk about open and closed ball. And this is what we're getting to next. And so when I'm talking about open ball-- but I'll try not to use that term here. But open would always mean this. Closed ball would mean that it's less than or equal to. Sorry, yeah?

AUDIENCE: Would you talk to me about the neighborhood example? Is that reverse-- is x in that case a subset?

PROFESSOR: Yeah, that's right, exactly. That's right. And I didn't actually-- I already posted the lecture notes. And so in the lecture notes, I didn't actually talk about tubular neighborhoods. But it maybe is worth thinking about. So you could look at it as-- again, you have a metric space here. And you could have a subset-- A has a subset, any subset, of the metric space. And you could define the tubular neighborhood of a subset. And so you could do this similarly. So this would be the y in x so that the distance from y to x here is strictly less than r for some x in a [INAUDIBLE].

So this is how-- so again, if you think about our tree, you take some-- I mean, maybe we shouldn't talk about-- maybe we should draw it more crazy, like some curve, maybe. And if you're looking at the set that have distance at most-- this is some sort of tube, like this. This here would be the [INAUDIBLE]. And then here, at the end, it would look like a cap, like this. This here would be like a tubular neighborhood of this curve. And so, as you're saying that in the case where A here is just a point, then those two coincide.

So now one of the key things when you're thinking about convergence in R, remember that this stuff, when we use this time and time again, that it was often-- we saw this for series, that it was often difficult to determine the exact limit of a series. But the question was always, does it converge? And so if it converged, if it converged, then often, I mean, also, if you would have using this in some of application, it's often in all kind of fields that it comes up that things are written in terms of a series.

And it's not clear what it's converting to. But you want to know whether it's converting. And then you also want to-- so you want to estimate what is the limit within a certain error term. And you can use some of these comparison tests to determine-- to estimate how close you are to the actual limit. But in any case, the thing is it's very useful to have a criteria that is easy to work with and that say that it's converging without having to be able to find the limits. And like the Cauchy sequence, this was like a notion of that.

And so again, the key on R, the key for using that was the Bolzano-Weierstrass theorem. So this was the one that was the key for using to get the quasi convergence theorem. So the Bolzano-Weierstrass theorem-- so remember that this is for any [INAUDIBLE] that any bounded sequence has a convergent subsequence. That was Bolzano-Weierstrass theorem.

In a general metric space, it makes sense to talk about a bounded sequence. But it's not always the case that a bounded sequence has a convergent subsequence. So it does not hold. The Bolzano does not hold. So warning, Bolzano-Weierstrass does not hold in a general metric space.

And in fact, of course, the example that we looked at, where the example was x here, like this, and when we're looking at that sequence, this is a bounded sequence. But it's bounded, but it doesn't have a convergent subsequence. It's actually a Cauchy sequence. But it doesn't [INAUDIBLE]. So it doesn't hold in a general metric space, but it does hold if the metric space is compact. And so that's what I want to talk about next.

But in order to talk about what it means for metric space to be compact, then I need to talk about-- I need to define what it means for a set to be open. So we need the notion of an open set, an open subset of a metric space.

So we have a metric space again. And so A here is said to be-- let's call it O for open. O is said to be an open subset if for all x in O-- so the idea is that you have here, this here is O. So this is maybe O.

And so the idea is that for each element, there exists some ball centered at that point. So for each x for all x in O, there exists. r greater than 0-- it may be incredibly small-- so that not just x but the whole ball around x is contained in the open set. So you see that not just this point here, but the whole ball here would have to be contained in O. So in this picture here, when I'm illustrating this, I'm thinking about x as being R2. And I'm thinking about the distance being the usual Euclidean distance.

But then, actually, if this is open, it means that you cannot include these points here on this curve. They would not be included in the set because if you take any point on this boundary here, so to speak, then a ball part of it would be outside the set. So if this really was open, then this curve here was not included. And you see that as you're getting closer and closer to that curve, you can still have a little ball that is inside the set. But as you're getting closer, the radius of this little ball have to be smaller. But that's fine.

So this is what it means for a subset to be open. And the first thing that you observe that with this definition-- there are always two open subsets. So again, this thing here-- so note that the empty set-- but there's no condition. If there was an empty set, say for all elements in the set, But there is no element, so there's nothing to be satisfied. So the empty set and all of x are open subsets. So do note that.

So in particular, if so if x here was not the whole real line but it was just the interval from 0 to 1, and then the distance between elements was the usual distance, then, actually, even the closed interval, which is the whole set, would be open. So you just should be careful with what the set is-- what set [INAUDIBLE].

So now the key here is that-- so if you have an open set, how do you construct a-- well, maybe the first thing I should do is I should prove a lemma that is the following. And then, after that, we will talk about how to get lots of open sets.

So we have two examples. At the moment, we have two examples of-- so if you take a metric space, at the moment, we know that there's always two subsets that are open, the empty set and x. But they're not very interesting. So a more interesting example is that if you take a metric space and you take a ball-- and now it's crucial how we define the ball. And so it's crucial that this inequality is strict. So for a metric space x here is in x. It could be any element in x. This thing here, r here, is strictly positive. It also has to be strictly positive.

Then this thing here is open. Any metric ball like this is an open subset. Let's try to prove that. And again, whether you're proving things like this for a metric space, the only nontrivial thing is the triangle inequality. So this is relying on the triangle inequality. And so for a lot of things, when you're proving things about a metric space, it's often quite helpful to draw a little picture. But of course, you have to be careful with, again, then the intuition comes from Rn. And sometimes, the metric is a little bit more complicated than that.

So this here, imagine that this is the ball of radius r. Here you have some x. Now take an element y. So suppose that y here is now in this ball here. Then we want to show that there exists that for some s bigger than 0, the ball here of radius s around y is now contained in this ball here. So here, you have y. And we want to prove that there is some little ball around y that is contained in the original ball. And so, again, this is going to rely on the triangle inequality.

So you have y here. y was supposed to be in this ball here. This means that the distance from x to y is strictly the s in r. You take now s here to be r minus this distance here. Why do I do this? In my picture here, I have here x. I have here y. And our intuition is that if we go from x and we go through y and then we go straight to the boundary, then this thing here is dx comma y. And this one here is r minus d. This is the remaining-- this whole here is r. This is dx comma y. So this here must be this here.

And so if you're setting this equal to s, and now you imagine taking a ball here, well, then at least on the picture, this whole ball is contained in the original ball. So let's try to see if that works. If that works, then it really only used the triangle inequality.

So we're setting s equal to this. And then we're looking at this ball here around S. And we take an element here in this ball. And we now need to prove-- so we need to show. We want to prove that this whole ball is contained in the original ball. So this means that we need to prove that for any such element set here is in this thing here. That's what we need to prove.

And so now you see-- so we take z. So we're going to look at this distance. In order to prove this, we need to prove that the distance from z to x is strictly less than r. So let's examine that distance. Triangle inequality, say, that this is bounded by the distance from x to y plus the distance from y to z. This one here from y to z lie in the ball of radius s around y.

So this means that this thing here, this thing here is strictly less than s. And so you see that this thing here is now less than d x, y plus s. Now you fill in what s is, this here. s was equal to r minus d x, y. That was the definition of s. And so now you see that this thing here is r because those two things cancel out. And you see that this here guarantee you that the inequality was strict. And so it really is that whole ball was-- so we Conclude that this whole ball here was contained in the original form.

And again, the proof of it was just from this picture that you make kind of rigorous. And the only nontrivial things at the moment we really have in a metric space is a triangle in concern because we haven't built up a lot of stuff. So you see, this immediately gives you a lot of example of open subsets of a general metric space. And if we had looked at exactly the same stuff-- tubular neighborhood-- we would find that those are also open subsets. Yeah?

AUDIENCE: Also we look at subsets, or do those have any value?

PROFESSOR: Close? Yeah, very much so. But we haven't defined what it means to be close. But you're right, and we will do this. So at the moment, we don't know what a close thing is. And maybe I will just define it. And then we can-- so let's define. But I'll talk more about this next time-- but now when you're asking. So we have x here is a metric space again. And then see here, it's a subset of x is said to be closed.

And then now it may-- seems almost cheating. It's said to be closed if the complement-- so this means that the complement of C-- so the complement of C-- these are all the x in x so that x is not in C.

So the complement a set is everything in x that is not in that set. And a set is said to be closed if the complement is open. And so now, once you have this definition and now prove this next time, that if you're looking at what I denote like this, so this is the elements where the distance to x is less than or equal to-- so this is not any more strict. This thing here is then a closed set.

AUDIENCE: Could you also define it by the way we define an open subset but saying that R instead of being strictly?

PROFESSOR: No, I couldn't define it like that. So I mean, you could define it other ways. There's another standard where you would talk about-- there's another equivalent to say that if you take-- but I don't want to get into that because-- but there's another way we get to this. And closed sets, we really mostly talk about next time. It was just because you were asking.

But let me just talk about-- so this is one, but I'll just repeat this in just a second. So in order to produce more sets, so we start with-- actually, I should just note, although just now when I talk about, just the definition, I should just immediately note that there are always two closed sets because you see, the empty set was always open, and the full set was always open. And so the complement of the empty set, that's everything. So that's closed.

And the complement of everything is nothing. It's empty set. So that's also closed. So those two sets are always closed. But let me just talk about-- so I want to think about how do you produce open sets? So how do you know that there are a lot of open sets? And so if you take sets, if you take-- so this is like operations on sets on subsets of a given set. So here, I'm thinking about just x as being a set. It does not need to be a metric space. So if I take a set then and I take two elements, two subsets. So these are two subsets.

Then I can form the union of these two. This is the union. And the union here of two sets-- so the union-- and again, I denote it like this, [INAUDIBLE], like this. These are the elements in x such that x is in u1 or x is in u2 or both. And so you often illustrate this with-- this is x. This is u1. This is u2-- u1 and u2.

Then this thing here is the union. And likewise, the intersection is things that is lying in both sets. So the intersection-- so this is, again, you have these two sets. And this is the intersection. And you denote it like this. And this is the x. So that x is in u1. And x is in u2. That's the intersection of these two sets.

And then there is a third-- and then we talk about generalization of this. So there's a third, and this will complement. But let me also introduce the notation. So if you take this subset of x, then the complement-- and there are slightly different ways of doing it of notation for it. And I always use this one because I think that's the clearest one. So this is like x take away u. And this is totally standard. There's another one that's often used where you have C. That's slightly less standard than this one. And sometimes, it can be confusing. This one can never be confusing.

And so this thing here is a compliment. And so this means that it's the x such that x does not belong to u. So that's standard operation of sets. But there is a generalization of this union here, where instead of just two sets, you could take potentially infinitely many sets. So if you take a family of subsets-- so alpha here is a family of subsets.

Then you could look at the union of the u alpha. This is denoted like this. And this is the x in x such that x is in u alpha for some alpha. I mean, it could be in many of the u alphas, but it has to be at least one. So this is the union of a family. And likewise, the intersection of a family is the x. And then it has to now lie in all of them. It's x in x, where x is in u alpha for all alpha. That's a union and the intersections of family.

And so now let me look at some example in particular. Let me look at an example of this. And then let me prove something about open sets. So let me look at an example here. Suppose that u alpha or u n-- I'm just going to look at-- it's going to be now a family of subsets of the real line. And they're going to be indexed by the natural number. So this thing here is going to be minus 1 over n to 1 over n.

And if I'm looking at the union of these guys-- but you see, this is a sequence of open sets. I mean, it's like for each n, you have an open set. But as n gets larger, they're smaller. The first one is the biggest one. So if you take the union of these guys, that just gives you the first one you started with. So this is the union of these. If you take the intersection of them, well, the only thing that is in all of these sets, that's actually the 0.

Now we need some stuff about operations of sets. And so there's some standard operations. And my recommendation is that not try to remember this. I don't remember it myself. So I always have to run through. But once you have done it once or twice, the proof is so trivial that it's not worth remembering. And I think the danger if you try to remember these things, you may remember it wrong. And that's much worse than just running through the proof.

So you have the following lemma here. So you have a set, and then you have subsets. Let's call it A, B, and then a family of subsets. And the thing is the first one is that if you take the complement of the complement, so if you take A and you take everything that is not in A-- so just think about it that if you take-- so here you have x. Here, you have A. This is A. Everything that is not in A, that's over here. But then if it's not over here, then it's A. So the first one is just that.

And actually, just because we're also running out of time, we really proved this. I mean, so if you have something that is not in A, then it's like here. But if it's not like here then it's like A there. That's that statement.

The second one is that if you take the complement of that, it's in the union of these guys. So if it's not, this just means that it's not in any of these guys. This here is the set. Imagine that you have something that looks like-- you have a bunch of sets like this. This is supposed to illustrate the A alphas. And if it's not this, just say that if it's not in any of these, well, then the claim is that, then, it's in the intersection of-- that it's not in this one, it's not in this one, and it's not in this one, and it's not in that one. So that's also clear enough. So this is the second one. Again, I actually wrote out the proof in the notes. So you can check that out.

And the third operation is that if you take x, and you are removing the intersection of this family-- so this means that it is not-- so this means that the intersection of the family in this picture here would be this thing here. And so it's not right there. Everything else is allowed. But that's the same. You see, that's the same as the union of x take away A. This here is the element that is not in one of-- there's at least one of the u alpha, A alpha, where it's not in, and so that's the same. So this is the same as this. Again, I wrote it out in the notes-- the proof, actually, of these things.

And let me just say-- maybe we can't quite I mean, we'll pick up in the notes where the other stuff-- but let me just say that when you prove of thing, and we even did this here when I'm just illustrating it with a picture. So we prove it using the following trivial lemma. And I wrote that out as a lemma in the notes, that if you are trying to prove that two sets are equal-- so suppose you're trying to prove that two sets are equal. Then, in a way, it's natural to divide this into two parts.

So this is the same as proving that A is contained in B and B is contained in A. So if you're trying to prove that two sets are equal, then you would prove that for any elements in A, it's also B, this is this thing. And you prove that for any element B, it's also in A. And so when I wrote out the proof of these three things, then I wrote it out like each of them has two parts that you prove that this set is contained in that and you prove that this is contained in that.

Now we're almost actually there. So now let me just say that-- so that's actually in one of the Pset problems. But there's no Pset. You have a little break now for Psets because I want to make sure that there is no Pset material we haven't yet covered in class, so that's why there's a break. And anyway, all of the Psets at this point are actually posted. So the last one I think is due-- I forget whether it's the 8 of May or something like that. And again, there's five left. You've already done [INAUDIBLE].

So now if x here, d is a metric space, then you have the following lemma that if O alpha are open subsets, then the union of the O alpha is also-- But this is quite trivial, as you see. I'll just indicate why. But it's actually left on a Pset. So the proof here-- but I will just illustrate it anyway that if you take an element here in the union, well, this means that there exists a value added. So it means that x is in O alpha for some alpha.

But this means that there is a ball around it because this here is open. It means that there's a little ball around it that is contained in this. But therefore, it's also contained in the union of these guys. And so this is really just a proof that the union of open sets are open.

We're out of time now, but next time, we'll talk about why if you take finitely many sets, why the intersection of finitely many open sets, why that is also open. But for intersection of open sets, you have to be careful because for union, you can take the union of any number of open sets. It's always open.

But if you take intersection u, in general, it's only true that finite intersections are open. And the example is one of the examples we looked at before where you take these sets here. These sets are open subsets. But if you take the intersection of all of them, then you just get 0. And that's not an open subset. So we're done.