[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: OK, so last time, we talked about-- so we had a function or sequence of function. We had an integral, say, I, and we had a sequence of function fn and a function f all defined on this interval. And then we talked about pointwise convergence.

So pointwise convergence means that for each-- so this is for each x when you're looking at this. So this is for each x fixed. This is a sequence. This is supposed to convert to f of x. So this was pointwise convergence. And then we also talked about uniform convergence. And this meant that for all epsilon greater than 0, there exists a capital N such that if little n is bigger or equal to this capital N, then the difference between fn of x and f of x, this here, is smaller than epsilon, and this here have to hold for all x.

So uniform, again, refer to that as uniform in x. The convergence is uniform in x. Then so we talk about that. And then one of the theorems we proved was that if you take a sequence, fn and [INAUDIBLE] was integrable, and these converge to some function f, then f is integrable. So this is uniform convergence. Then f here is integrable and the integral is the limit of the integrals. So the integral from a to b of fn x dx is equal as the limit here of this. So the limit exists and it is equal to the integral of f.

So this is how uniform convergence and integrals interact. Suppose you take a sequence here. So this is a theorem. Suppose you take a sequence fn of functions on some interval into R. You have x0 is in this interval. And then you have the following if you have the following properties, that one is that this sequence here converged to some number C.

So the sequence converged to some number c. Two is that these functions here, fn, are differentiable. So these are maybe, I'll just say here, differentiable. And these here converge to some function g. And then 3 is that these functions here-- sorry the derivative converge, and I'm also requiring that the derivative are continuous.

Suppose we have these three things. So we have the sequence of differentiable functions. If you just fix this one particular point, x0, then the values of these functions converge. And then the functions-- their derivatives are continuous, and the derivative converges to some function. And this convergence here is uniform.

Then the claim is that then there exists f on the interval so that f is differentiable. The derivative here is continuous. And you have the following two properties that the function here-- these functions converge to f uniformly. I'm going to restrict my interval, which would take it just to be some interval from A to B. And I want it to be bounded, like that.

And so these converge uniformly, and two is that their derivative converges to the derivative of f uniformly. So we want to prove that. And then so we're going to define a function f. So g here is a proof of this. Since this sequence here-- since these are continuous and these functions here convert to g uniformly, then this means that then, therefore, g here is also continuous. So we proved this already, that if you have a sequence of continuous function that converges uniformly, then the limit is also continuous.

And so then we can look at-- so we can therefore define f as a function on this interval here by that f of x is equal to the integral from x0. So I'm defining it, sorry, to be the c constant, c plus the integral from 0 to x of g of x, g of s ds. So I define this function like this. g is continuous, so this is a nice continuous function. And in fact, by the fundamental theorem of calculus, f here is differentiable with the derivative equal to g.

We also have that we also have that if we're looking at f of x0, fn of x0, then we know already that these converts to c. This is the assumption one up there. But this here is the value of f at x0. So we have that.

I want to now prove that not just at this point but that this sequence converges to f uniformly. b we get for free. So b follows from that this fn prime converges to g. And this here is the derivative of f. And we have this uniformly by assumption two.

So all we need to show is-- so we just need to show that each fn converges to f uniformly. We just need to show that the property a-- and so we're looking at fn of x. We have to look at this difference. And for a given epsilon we have to find at little n. So we have to find a capital N so that if little n is bigger than this capital N, then this difference here is smaller than epsilon for all x.

But let's try to estimate this here first. So this difference is f. So we want to estimate this difference. And we can use the fundamental theorem of calculus or use the fundamental theorem of calculus to write f of x. We can write this as the value at x0 plus the integral of the derivative from this point. And then this should be less like this. We can use the fundamental theorem of calculus to just rewrite fn like that.

And so now I want to look at this difference here. And so to look at this difference fn of dx minus f of x, now I can fill in this form of it. So I have that this is f of x0 plus the integral from x0 to f prime s ds. And then I have to subtract f of x. But f of x was given by this formula here, and so minus c plus this integral here, g of s ds, like that.

And so now I'm just rearranging terms. So I just think about this thing here as fn of x0 minus c. I take these two first, and then plus the integral of x0 to x f prime n s ds minus this integral here, like that.

And then I use the triangle inequality that this thing here is less than fn of x minus c-- sorry fn of x0 minus c, absolute value, like this. And then I have the second integral here. So the absolute value or the difference here, if you do it slowly like this, so you have this minus this in here of g of s ds, like that.

Now I can think about this thing here as I can find the integral, so I can write this thing here as-- so I'm just keeping this first term here. And then the second here I can think about this as integral of the difference of this function. And this here was an n here, like that.

So now, here, I can just take the absolute value sign. If I take the value I and under it, then I get something potentially larger. So this thing here is less than or equal to f of x minus 0 minus c. That's the first term. And now I'm replacing it by something that is bigger or equal to this, so where I'm just integrating from x0 to x, and then of this difference here, f prime s minus g of s ds.

And now I'm using that the fn-- so actually, let me just write this thing here. So I can write this thing here. This one here, I'm keeping. This one here, I'm keeping. So I'm writing this as less than or equal to keeping the first term. And the second term, you see, this thing here is this difference between these two. Well, the difference is at some s is smaller than the maximum for all s.

But the maximum for all s, that's what we call d fn prime g. That was the maximum over all s. And so now we just have-- so this function in here is less than this. And now we are using-- this is where I want to use that the interval is bounded. So I just get plus b minus a. This is the length of the interval, like that.

And so now you see that s-- and now there's no reference to x on this side. And so you see that, as n goes to infinity, well, this thing here goes to 0 and this thing here goes to 0, so as this upper bound goes to 0. And again, note that there's no reference to x. There's a reference to x0. But that's just some fixed point. So this is uniform in x. So this implies uniform in x converges.

And so I want to draw some consequences of this. First, let me look at what will be, shortly, a particular case of a more general phenomenon. So this example we looked at several times last time. But it's now applied in this case here.

So suppose you're looking at the functions En of x. The En is where you're summing from k equal to 0 to n of x to the power k divided by k factorial. I looked at those last time, these functions. And we call them En. And then we also looked at the exponential function and the exponential function in terms of power series. That was just this power series.

And we have that we have that En here so that for all-- we used Weinstrass M-test to-- so by Weinstrass M-test for all L-- L is a positive real number-- we had that these functions converge to E uniformly on an interval like this. They don't necessarily-- in fact, they don't. But the claim here about uniform convergence is just on every bounded interval. So you fix a bounded interval like that interval, and then you have uniform convergence. It's not like uniform convergence of all of R at the same time.

So we have that, and now if we are looking at-- so now if we're looking at the function-- if we take now the derivative of the En's-- so what is the derivative of the En's? So the derivative of the En-- but the En is just a polynomial. So you can just differentiate each term on its own.

So this here is the sum here of x to the power k minus 1 divided by k factorial. And here, you would be summing from k equal to 0 to n. And now I can write this also. You see, because when k is equal to 0, the first term doesn't really count. So this thing here, I can write as-- I mean, it doesn't count because it's 0. So you don't really need to write that term. So you can just write it from k equal to 1 to n. And then you can write it again as k times x to the power k minus 1 over k factorial.

But k factorial is just-- it's the same as k times k minus 1 factorial. So this means that k factorial over k is just the same as k minus 1 factorial. Or, if you will, if I take the inverse of this, k over k factorial is the same as 1 over k minus 1 factorial. It's just flipping the denominator and the denominator. So this finite sum I can write as k equal to 1 to n because you see, this thing here is exactly this. So we can just replace it. You can write it like that.

But now this thing here, I'm summing from 1 to n. The first term, I get that is just this here by definition when k is equal to 1. That's 0 factorial. 0 factorial, by definition, is 1. So this thing here and this thing here, by definition, is also 1. So this is just 1. So starting off at 1, when k is equal to 2, then this becomes x. This will become 1 factorial. 1 factorial is 1. So this here is x. So this here is like this x.

The next one, when k is equal to 3, then you get x squared. So you get x squared. And down here, you get 2 factorial. And so you continue like this. And the last one you get is x to the power n minus 1 over n minus 1 factorial. Let me just say-- so you see that this thing here is nothing but En minus 1 of x. So the derivative of the En is the previous element En minus 1. Yeah?

AUDIENCE: I think that cleared it up.

TOBIAS COLDING: OK. So now we have that these En, if I'm looking at the En, is 0. The value here of these guys-- the only term that matters is the first term, and that's 1. So this here for each n.

So now what you have is-- so now we have that you have that if you're looking at this sequence, you have this. We have that the En prime is equal to En minus 1. And we already proved that these here-- and it doesn't matter whether you're looking at the previous one-- that they converge to E uniformly. This is a uniform convergence on this interval here. And then obviously, 3-- obviously, the En prime here, they are polynomial, so continuous.

So therefore, by the theorem, there exists some function f So that these are converged to this uniformly. That's no great surprise. And we also have that this thing here, the derivative converges to the derivative of this here uniformly. And again, this is on integrals of this form. Well, we already know that it converges uniformly. This is just by-- I mean, it can only converge to one function because uniform convergence means that it's converging pointwise at each x and pointwise convergence, then the limit is just one thing. So we already know that this here is E of x.

So you see that the conclusion here we get is now that this and this here is, as we said, En minus 1. So the conclusion that we get is that the E-- so this is the conclusion from the theorem up there-- that these En minus 1 converge uniformly to the derivative of this function.

The theorem said that if you have these three conditions, then there exists a function f that is differentiable and the derivative of it is continuous. And you have these two properties. But we already know that these converge to this. So we've already identified what this function is. So the theorem just gave us that this function must be differentiable and the derivative continuous.

And now we try to figure out what that derivative is. But the derivative of these functions are the same functions we started with, but just shifted one backwards. So these here, also, they converge to this. That's what the theorem say. But the En minus 1, they also converge to E. So this means that these two functions here, this here, and this here must be the same. So this means that E is differentiable and the derivative is itself. Yeah?

AUDIENCE: So we defined this for all n greater than sum--

TOBIAS COLDING: This here was for all-- this here was defined for any nonnegative integer. And so I mean, when I'm talking about down here, then I'm thinking about n bigger or equal to 1 because I didn't strictly speak define E minus 1.

So now I want to do this more generally. So I take a power series. Also, by the way, the plan-- so after these lectures, there's two more lectures, Thursday and Tuesday next week. And I already updated the course plan to reflect this. So my plan is next time, I will prove existence of uniqueness of some ODE using things we have learned here about metric spaces. And then, Tuesday next week, we'll do a review of the class. So there won't be any new material on Tuesday.

Also, for whatever it's worth, I decided in the end with a friend to turn in the lecture notes, to turn it into a book, just because the TPBB that was like the main reference, I didn't find that great anyway. It was also 1,000 pages, who would want to read 1,000 pages. And so the course notes here is shorter, as you probably have seen, and trying to focus on the key points. So eventually, there will be a book.

So now I want to do this more generally. So we take a power series. So we have an as a sequence, and we're looking at a power series like this. So this is a power series. And so just to remind you, the radius of convergence is given as follows. So we take M, and M is defined to be the lim sup as n goes to infinity of the n-th root of absolute value of an.

And then the radius of convergence, so R here, is then 1 over M with this usual convention that if M is infinity, this is 0. If M is 0, this is infinity. And so then we have, again, that we have that-- so we have that the series here, so if you're looking at this k equal to 0 up to n of an xn, this here converts to this limit here uniformly on each interval of this form here, where L here is strictly less than 1. And this is, again, is Weinstrass M-test.

Now what I'm interested in is I'm interested in. So I want to be able to calculate the derivative of this. So we define a function f by that f of x is equal to the sum here from k equal to 0 to infinity. So this is the power series x so k, like that. And we want a formula for the derivative. And in fact, I want the formula for any derivative of-- so say the M derivative of this function.

So now I look at-- so we had take this power series, and I'm looking at the power series that is given by-- so suppose you're looking at the power series where for each term, so you have in the original power series. It's an times x to the n. Then I'm looking at n an times x minus 1 like this. And here, I'm summing from 1-- I mean, it doesn't matter-- sum from 0, so 0 to infinity, like that.

Now the radius of convergence for this-- so what is the radius of convergence? The radius of convergence for this is the n-th root of n times a to the n, like this. And this thing here is the same as n to the power 1 over n times the n-th root of an, like this.

Now this thing here, if you're looking at n like this, this here is-- you can write this thing here as E, and then it's log n over n. This here, as n goes to infinity, converges to 0. So this thing here converges to 1 by E to the 1. And so you have that lim sup here. So the lim sup of the n-th root of n times an, like this-- this thing here is because this here, you can write this as the same as lim sup. n goes to infinity. n goes to infinity.

It's the same as the n-th root of this thing here times the n-th root of-- like this. This here converts to 1. So it really just converges-- so it's equal to the lim sup as n goes to infinity of the n-th root of n. And that was what we call M originally. So this lim sup is the same as this lim sup. So this means that M for this new sequence is the same as M for the old sequence. So this means that the radius of convergence for the new sequence is equal to the radius of convergence from the old sequence.

So if you take the derivative term by term and you form this new power series, then it has the same radius of convergence as the original passes. And so now you can iterate this. So you can iterate this. So you can take the-- so this is just taking the derivative term by term one time. If you now take the derivative, suppose you're starting with this sequence here. So you're starting with this power series. And you're taking the derivative term by term. Then you get as we did up there, you're getting this power series. So now you take a derivative again term by term. So you're getting this right.

And the thing is that each time the radius of convergence for this same radius of convergence, these two here-- same radius of convergence-- it's because of the same principle, same radius, and so on. And so you can continue. And if you do this, if you do this M time, then you're getting n-- so if you do this M times, then you get that this thing here is x minus M, and you're getting an. And here, you're getting n minus 1, n minus 2, up to n minus M plus 1. So you're getting that. So that is after-- this is when you do it n times. And so again, these here will have the same radius of convergence.

Now this thing here, if I want to write this a little bit more compactly, if I thinking about this thing here, n minus 2 up to n minus M plus 1, I can write this thing here as n factorial. But then I'm getting all these-- so if I wrote it as n factorial, then I would also have-- it would keep going like this. And the next one here would be n minus M. So if I write it-- so I can write this thing here times this factorial is equal to that. Yeah? Sorry.

AUDIENCE: [INAUDIBLE]

TOBIAS COLDING: So I think you were complaining about one thing that I sort of pushed under the rug. And that is the following, that if you take the way we defined-- so if you take a power series, again, like this, and you define the radius of convergence, then you define it from this n element. I think that's what you're complaining about. And so you take the coefficient to the n power. That's not exactly what I did. Is that what you're complaining about? And that doesn't matter.

And the thing is so when you define the radius of convergence, you took this n power, like this. But there was a coefficient to where this here was to the n.

When I'm doing it over here, there's two ways of dealing with it. One is the one you suggest, and the other one-- and that was why I put it on slipped it under the rock. Is that that's the way the radius this with the radius of convergence, it comes from comparing it with the geometric series. And then it really doesn't matter for this term in terms of x whether you have shifted it or not.

But if you do it, if you do it as you're suggesting, which would be totally fine, so then this thing here, what you really should have done is that we should have thought about this n, an, and then we should take this here, and instead of the n root, we should take the n minus 1 root. But that doesn't matter either. That's going to be the same also. So either way, you can check either of these two things. And I was thinking about the first and where it's coming from.

Sorry, now I lost my thread here. So I was just observing that if you can write the n factorial as first taking all of these factors and then taking n minus 1, so this means that n factorial divided by n minus 1, so n minus m factorial, this is equal to n minus 1, n minus 2, down to n minus m plus 1, like that.

But you see, that's exactly what we have done here. So this thing here, so what I conclude from this is that if I take a power series like this, and I take the derivative term by term, and I do this M times, then this is what I get is n factorial n minus M factorial and then an xn n minus M.

So this is when we do it term by term, taking the derivative. And we already have that these here have the same radius of convergence.

And so now we have the following theorem. Then if you take a power series, so you take a power series like this, R is the radius of convergence. And if L here is strictly less than some positive number but strictly less than the radius of convergence, then these guys here converge to-- so when you're summing, when you take a finite sum, these converge to this uniformly, but not only these.

But not only that, but you also have that-- more interesting-- you have that this sequence here, k minus M-- k factorial divided by k minus M factorial ak xk minus M, that these sequence here converge uniformly to the derivative. This here is f of x to the M derivative of x of f. So we have that these functions-- but that we already proved that these functions converge, and their derivative here converge uniformly, like this, on the same interval.

Now why is this the case? So this here is again it's following. So this follows the proof-- follows from the same theorem-- follows from the theorem we proved earlier, we showed earlier. And let's just go through that. So let me go through it.

So let's assume, first, that M is equal to 1. So I just take one derivative-- well, then I'm looking at this sequence here, k xk k equal to 0 to n. I'm looking at these that converge to this infinite-- to converge to f. f, again, is this infinite power series.

And I'm looking at the derivative of these guys. So these are these. And I know now that these here converge uniformly and these here converge uniformly the power series-- that is, this infinite series. This here was the finite part of it. And this here is the infinite part-- the whole thing. So we have that.

This means, in particular-- so this thing here implies, of course, that when you're looking at 0, when you're looking at 0, if you evaluate this thing here at 0, then, in this one here, you get in this is the sequence of functions, f of x from before, these are the fn prime of x.

And so now I claim that-- so the theorem that we proved earlier is that if you take a sequence of functions that are differentiable and the derivative is continuous-- these are all polynomials, so this is definitely the case. And we know that these values here fn-- in fact, at 0, they're equal to f of 0. So we had three conditions. We had that the fn in some particular point that we call x0, this is now 0, that this converged. We have that the f prime was converging uniformly. Well, we know that these converge uniformly. And then we had that the f prime should be continuous. But they were obviously continuous. They were polynomials.

Then we had that the derivative converged uniformly and that the function that they converge to-- the derivative converge and also the original function converge and that the derivative converge uniformly to the derivative of the function that we converge to. So you see that this is just checking that we have the three property from the theorem before. And then we get the two conclusion. And the one conclusion that we're interested in is that this here is then the power series for the derivative of f.

And then you can do the same thing with-- so let me just say that you can do the same thing if you are integrating. So here, we proved-- so suppose you take a sequence here that is converging to f. Well, suppose you take a power series here. Suppose you take this power series, ak xk k equal to 0. So this is the finite-- this is the power series, sorry.

And suppose I'm interested in what is the integral from a to b of this power series f of x dx. And here, I'm assuming that a and b are in the interval, where the power series is convergent. So this means, of course, that if you think about it that you have here 0. You have R. You have here minus R. And then a and b is lying-- and so this means that this whole interval is lying comfortably inside some interval like this.

And I'm asking what is the nice formula for this integral here? And so then, well, I can use not the theorem we proved earlier but the theorem we proved last time. And so what is that? So if I define fn of x to be equal to the sum here from k equals to 0 to n of ak xk-- sorry, x to the power k-- this thing here, these here converge uniformly to f of x on any interval like this when L here is strictly less than the radius of convergence. So they converge here uniformly.

But the theorem we proved earlier was that if you take a uniform convergence like this and you take a and b in this interval, then the integral of these guys here converge to the integral like this. You have this convergence here.

But these here are easy to integrate. But because they're polynomial, so this means that the fn s, the fn of s, when you integrate this from a to b ds, these guys are the sum here of ak sk k equal to 0 to n, and then the integral here, ds from a to b because it's just that polynomial.

And so for polynomial, well, we can just use that it's a sum of these guys, ak sk ds k equals to 0 to n. And this thing here, when we integrate this, that's just the-- I mean, now it doesn't get much nicer unless you make one of them equal to 0. So you have this. So let me say that if a here was equal to 0, and B was equal to x, so if that's the case and you're looking at this integral from 0 to x of fn s ds, we know that this here converged to the integral of the power series.

On the other hand, this integral here-- on the other hand, this integral here, if you're looking at the sum here and then sum, and now the a is 0, b is x, so then is this here ak xk, now when we integrate this term here, we're getting sum here k equal to 0 to n. And then this here becomes ak k plus 1 x to the power k plus 1.

And so you see that if you're looking at-- so you see that these guys here now converge to the integral from 0 to x of f of s ds. That's what we get from the theorem. So the conclusion here is that if you take a power series and you're looking inside the radius of convergence, then the power series is differentiable as many times as you want. And the derivative is given by a power series. And it's just the derivative of the individual term by term.

And likewise, if you want to integrate a power series, then the integral, at least when you're integrating because, otherwise, the formula is not quite as nice, but you do get a formula, but it just doesn't look so nice. But the integral of a power series, you can just integrate it term by term.

So now the next thing we want to talk about-- and I'm just trying to set up for next lecture. And so we want to talk about differential equations and how we're using some of the stuff that we've proven in this class to solve differential equation. And it's an example where it may not be very obvious that you could use metric spaces to solve a differential equation. But this is nevertheless the case. So we're looking at a differential equation.

So differential equation is an equation is an equation in-- an unknown in an unknown function and its derivative. So let me give you some examples of this that you take-- you're looking at a function f. You're trying to find a function f so that-- so this is like the first example. You're looking for a function f so that the derivative is x.

Well, we already know that this here, the solutions to this f of x equal to 1/2 x squared plus some constant. The constant when you take derivative, you get 0. And the derivative of this thing here gives you x. And those are the only solutions.

Another example here is where you're taking something that looks like f prime minus f squared, so f of x, like this. And if you're looking at the function here, f of x is equal to 1 minus 1 over x. If you take this function here and now you take the derivative, so now I'm using the chain rule, so I get 1 minus 1, like this, squared.

And then I take the derivative of this. So that's minus 1. So that's minus 1 times minus 1. So this here is minus 1 minus x squared. And so you see that when you take this function here, the minus sign-- I think it should be a plus. So when you take-- no, no, no, it should be minus. No, no, that's right, sorry, yeah, thanks. This is wonderful. This should have been a minus like that. And so this here is now is indeed plus. And then it's this equation.

The next example-- so again, a differential equation is something that involved an unknown function and its derivative. So suppose you take something that looks like f, f prime, f double prime equal to 0. If you're taking the function f equal to some constant, this is a solution. And if you're taking f equal to, say, x or some constant times x, this is also a solution. They're both solutions. And this is just because the second-- in this case, the second derivative is 0.

So now what we will be interested in is we're going to be interested in what's called an ordinary differential equation. And we only looked at this. We only looked at function of just one variable. So this is what the ordinary referred to. So we want to look at ordinary differential equation. And an ordinary differential equation is abbreviated by ODE.

And if it's not an ordinary, then it's usually a partial differential equation, which involve a function that is of more than one variable. So it will involve what's called the partial derivative of the functions.

So the unknown function will be denoted by y. And the equation that we're looking at is that we want some particular value at, say, 0. And then we want the derivative satisfying an equation. And the equation that we're looking for is that we have-- so f here and g are given. So think about those as functions that are given. Those are two given functions. But they may not be super nice. But you know that f is differentiable and the derivative is continuous and g here is continuous. But in principle, you know they are given.

And now you're looking for-- but it may be very, very complicated. Maybe f is sine or something like that. And so you're looking for an unknown function y so that the derivative-- so you think about y as a function of x. And the derivative of y is supposed to be equal to f but is now applied to this unknown function and then plus g of x. So this here is much more tame, but this is a bit complicated because it involves the unknown function.

And so we want to see-- so we are asking the question, does this ODE have a solution? And is the solution unique? So we're looking for a function. So here's what we're looking for. We're looking for a function y defined on some interval into R. This function has to be differentiable. So this has to exist. A 0 has to be in that interval because a is just a constant. And then we want, obviously, y to satisfy this equation? That's what we're looking for.

So this is existence. The thing is that what we will prove is that there exists some interval-- so we will prove the following. So we prove the following theorem, which has many different names, and it is attributed to various people. But it's I guess. it's most often named the Picard-Lindelöf theorem. And so this is that given f and g as above, so f has to be differentiable and the derivative continuous g just has to be continuous. Then for any a, there exists a delta greater than 0 such that-- and a solution to this differential equation is ODE on the interval from minus delta to delta.

So we proved that there exists a solution near the origin. The thing is that a solution may blow up. So it may not exist on a very long interval. It may just exist on a small interval. It's all depending on f and g. But in particular, will depend on f. And that moreover, it is unique, meaning that if you take another solution defined on the same interval, then it must be that solution.

And so now I just have a minute to-- but we will prove this next time anyway. But I just want to, as a teaser, if you will, I will just say that the proof of this uses the contraction mapping theorem. Maybe I will remind you what it is.

So recall that the contraction mapping theorem is the following-- that suppose you take a metric space x and d-- so D is the metric, x is the space-- and suppose you take a map from x to itself. So then T is a map from x to itself. T is said to be a contracting map.

If there exists some constant, that constant-- that's key. It has to be strictly less than 1 so that if you measure distances between the images, so for any x and y in capital X-- this is for all x and y in capital X. If you look at the distance between the images of these two points, then it's less than or equal to this constant that is strictly less than 1, like this. So that's a contracting map.

And the contracting mapping theorem that we prove, I'll remind you about it again next time, but just the contracting mapping theorem that we proved is that-- so we proved the following theorem, which is a contracting mapping theorem, that if x here is Cauchy complete, and T from x to x is a contracting map, then T has a unique. fixed point. So there exists a unique one-- often write like this-- there exists one and only one x0 so that this map takes x0 to itself. So that's a contraction mapping theorem.

And so it seems like what does ODE has to do with them, this abstract concept of a metric space? So it's tied together using this. And then the space x, it's not-- you may think that that's close, but it's not quite the case. You may think that it's like a continuous function on an interval like this. That's close. You may think that because that's a space we've talked about a lot, this here, with this metric d. We talked about with this maximum distance. It's not actually quite the case.

It's a little bit more complicated than this. But x is related to this because x here will be a ball around the function that is constant, the function that is constant a in this space. And we have to choose this R carefully. And so that's a space x. And now I haven't defined the function the map g. But we'll do that next time. And I will remind you about this also. Any questions?