SPRING 2025 - 18.100B/18.1002

TOBIAS HOLCK COLDING

Lecture 1

Two key topics for this class:

- How to write a mathematical proof.
- How to prove theorems.

Here is an example:

Intermediate value theorem:

- Suppose that $f : [a, b] \to \mathbf{R}$ is a continuous functions.
- Assume that $f(a) < 0$ and $f(b) > 0$.

The intermediate value theorem says that there exists a c between a and b where $f(c) = 0$.

- How do we prove this?
- If we draw a picture, then it seems obvious, but how to we actually prove this?

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That a function is continuous basically means that when you draw the graph of the function the pencil is not allowed to leave the paper.

- How do we make this into a proper proof?
- What properties of the real values are needed for a proof?

This leads to several questions:

- Q1: What is a real number?
- Q2: Why is $\sqrt{2}$ a real number?
- Q3: What is $\sqrt{2}$?

The answer to these questions: $\mathbf{R}$ is a complete ordered field that contains the rational numbers $\mathbf{Q}$.

Here is some notation:

(1) $\mathbf{N}$ is the natural numbers. This means that $\mathbf{N} = \{1, 2, 3, \cdots\}$.
(2) $\mathbf{Z}$ is the integers. This means that $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$.
(3) $\mathbf{Q}$ is the rational numbers. So all numbers of the form $\frac{m}{n}$, where $m \in \mathbf{Z}$ and $n \in \mathbf{N}$.

Properties:

Rational numbers:
Rational numbers $\mathbf{Q}$ are numbers of the form $\frac{m}{n}$, where $m \in \mathbf{Z}$ and $n \in \mathbf{N}$.

(1) When are two numbers the same?
$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \iff m_1\, n_2 = m_2\, n_1\,.$$
(2) How do we add two numbers?
$$\frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1\, n_2 + m_2\, n_1}{n_1\, n_2}\,.$$
(3) How do we multiply two numbers?
$$\frac{m_1}{n_1}\,\frac{m_2}{n_2} = \frac{m_1\, m_2}{n_1\, n_2}\,.$$
(4) When is one number less than another?
$$\frac{m_1}{n_1} < \frac{m_2}{n_2} \iff m_1\, n_2 < m_2\, n_1\,.$$

For this to make sense we need (for instance) to show that multiplication is well-defined:

This means that if we have two representations of the same rational number
$$\frac{m_1}{n_1} = \frac{m_2}{n_2}$$
and likewise
$$\frac{p_1}{q_1} = \frac{p_2}{q_2}\,,$$
then
$$\frac{m_1}{n_1}\,\frac{p_1}{q_1} = \frac{m_2}{n_2}\,\frac{p_2}{q_2}\,.$$
Proof. We have that $m_1\, n_2 = m_2\, n_1$ and $p_1\, q_2 = p_2\, q_1$. Therefore,
$$m_1\, p_1\, n_2\, q_2 = m_1\, n_2\, p_1\, q_2 = m_2\, n_1\, p_2\, q_1\,.$$
□

This illustrate how detailed a proof should be.

A Field:

Definition:
A Field $\mathbf{F}$ is a set with two operations that we are denoting suggestively by "+" and " $\cdot$ ". Those two operations satisfies the following axioms:

Additive properties:

(1) $x,\, y \in \mathbf{F}$, then $x + y \in \mathbf{F}$.
(2) $x + y = y + x$.
(3) $(x + y) + z = x + (y + z)$.
(4) There exists an element $0 \in \mathbf{F}$ such that $0 + x = x$ for all $x \in \mathbf{F}$.
(5) For all $x \in \mathbf{F}$ there exists an element, suggestively, denoted by $(-x)$ such that $x + (-x) = 0$.

Multiplicative properties:

(1) $x,\, y \in \mathbf{F}$, then $x\, y \in \mathbf{F}$.
(2) $x\, y = y\, x$.
(3) $(x\, y)\, z = x\, (y\, z)$.
(4) There exists an element, suggestively, denoted by 1 such that $1\, x = x$ for all $x \in \mathbf{F}$.
(5) For all $x \in \mathbf{F} \setminus \{0\}$ there exists an element, suggestively, denoted by $\frac{1}{x}$ such that $x\, \frac{1}{x} = 1$.

The final axion that we need is an axiom that chains addition and multiplication together:

- $$(x + y)\, z = x\, z + y\, z\, .$$

Theorem: For any field 'zero' is unique.

Proof. Suppose there are two. Let us denote them by 0_1 and 0_2. Then

$$0_1 + 0_2 = 0_2$$

since 0_1 is a 'zero' and

$$0_1 + 0_2 = 0_1$$

since 0_2 is a 'zero' so $0_1 = 0_2$. □

Examples: $\mathbf{Q}$ is a Field, whereas $\mathbf{N}$ and $\mathbf{Z}$ are not Fields.

Ordered set: An ordered set $\mathbf{S}$ is a set with a relation $<$ with the following properties:

(1) For an $x,\, y \in \mathbf{S}$, one of the following holds: $x < y$ or $y < x$ or $x = y$.
(2) If $x,\, y,\, z \in \mathbf{S}$ with $x < y$ and $y < z$, then $x < z$.

Ordered Field:
An ordered Field is an ordered set that is also Field and has the following two additional properties that chains the operations in the Field together with the ordering:

(1) If $x < y$, then $x + z < y + z$.
(2) If $x > 0$ and $y > 0$, then $x\, y > 0$.

Example: $\mathbf{Q}$ is an ordered Field.

Theorem: If $x < y$ and $z > 0$, then $x\, z < y\, z$.

Proof. We need to show that $x\, z < y\, z$ or equivalently $y\, z - x\, z > 0$. The latter can be rewritten as $y\, z - x\, z = (y - x)\, z$. Since $y > x$ we have that $y - x > 0$ and the claim therefore follows since $z > 0$. $\square$

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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MIT, Dept. of Math., 77 Massachusetts Avenue, Cambridge, MA 02139-4307.