SPRING 2025 - 18.100B/18.1002

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Lecture 2

The real numbers $\mathbf{R}$ is a complete ordered Field that contains $\mathbf{Q}$.

Question: What is the difference between $\mathbf{R}$ and $\mathbf{Q}$?

One difference is that $\mathbf{R}$ contains $\sqrt{2}$ and $\mathbf{Q}$ does not.

$\sqrt{2}$ is a number x so that $x > 0$ and $x^2 = 2$.

Theorem: There does not exists a rational number x so that $x^2 = 2$.

Proof. We will argue by contradiction. So suppose that there exists a rational number $x = \frac{m}{n}$, where $m \in \mathbf{Z}$ and $n \in \mathbf{N}$, so that $x^2 = \frac{m^2}{n^2} = 2$. We can assume m and n does not have a common factor (other than one). We have that $m^2 = 2\,n^2$ and so 2 is a factor in m^2 and therefore in m itself. This means that $m = 2\,m_1$, where m_1 is also an integer. It follows that $m^2 = 4\,m_1^2 = 2\,n^2$ and therefore $2\,m_1 = n$ and so n is also even. We have now that both m and n are even and so have 2 as a common factor. This is the desired contradiction. This show that there is no rational number x with the property that $x^2 = 2$. $\square$

How do we add $\sqrt{2}$ to the number system?

$\sqrt{2} = 1.4142136\cdots$

So 1, 1.4, 1.41, 1.414, 1.4142, 1.41421, $\to \sqrt{2}$.

$\sqrt{2}$ is the limit of a sequence of numbers.

Completeness of $\mathbf{R}$. (Least upper bound property.)

Completeness is: If a subset A of $\mathbf{R}$ has an upper bound, then A has a least upper bound.

Suppose that $\mathbf{S}$ is an ordered set and A is a subset of $\mathbf{S}$, then M is an upper bound for A if for all $a \in A$ we have that $a \leq M$.

Example: If $A = \{1, 2, 3\} \subset \mathbf{Z}$, then 4 is an upper bound, whereas 2 is not an upper bound.

Example: If $\mathbf{S} = \mathbf{Q}$, then $\mathbf{N}$ as a subset does not have an upper bound (we will return to this shortly).

Least upper bound: Suppose that $\mathbf{S}$ is an ordered set and A is a subset that has an upper bound. We say that M is a least upper bound for A if M is an upper bound for A and for any other upper bound M_1 we have that $M \leq M_1$.

Complete ordered set: We say that an ordered set is complete if any subset that has an upper bound has a least upper bound.

Theorem: There exists a complete ordered Field that contains $\mathbf{Q}$.

This Field is denoted by $\mathbf{R}$.

We will not prove this, as a proof would take us too far a field, rather we will take it for granted.

Theorem: $\sqrt{2} \in \mathbf{R}$.

Proof. Let $A = (0, \sqrt{2}) \cap \mathbf{Q}$. That is A consists of all the positive rational numbers a so that $a^2 < 2$. Let x be the least upper bound for A. Note that A is nonempty (since $1 \in A$) and that 2 is an upper bound for A. Note also that $x \geq 1 > 0$ since it is an upper bound. We need to show that $x^2 = 2$.

We will first show that $x^2 \leq 2$. Suppose not; so assume that $x^2 > 2$. We will show that this leads to a contradiction. Consider

$$(x - h)^2 = x^2 - 2\,x\,h + h^2 > x^2 - 2\,h\,x\,.$$

As long as $h > 0$ is chosen so that

$$2\,h\,x < x^2 - 2$$

or, equivalently, that

$$h < \frac{x^2 - 2}{2\,x}$$

then

$$(x-h)^2 > 2$$

and therefore $x - h$ is also an upper bound for A. This contradict that x is the least upper bound. We therefore have that if x is the least upper bound for A, then $x^2 \leq 2$.

To show the reverse inequality (that $x^2 \geq 2$) we argue similarly. Assume that for the least upper bound x we have that $x^2 < 2$. Consider $x + h$, where $0 < h < 1$. We have that

$$(x+h)^2 = x^2 + 2\,x\,h + h^2 < x^2 + 2\,x\,h + h = x^2 + h\,(2\,x+1)\,.$$

Since we are assuming that $x^2 < 2$ we can choose h positive so that

$$h < \frac{2 - x^2}{2\,x+1}\,.$$

We therefore have that

$$(x+h)^2 < x^2 + 2 - x^2 < 2\,.$$

This is the desired contradiction and show that $x^2 \geq 2$. Together with the first step we have that $x^2 = 2$. □

Corollary: $\mathbf{Q}$ is not complete.

Proof. If $\mathbf{Q}$ was complete, then $\sqrt{2} \in \mathbf{Q}$ but we have already proven that there is no rational number with the property that $x^2 = 2$. □

Archimedean property: For all $x \in \mathbf{R}$, there exists a natural $n \in \mathbf{N}$ so that $x < n$.

Proof. If this was not the case, then $\mathbf{N}$ would be bounded. To see that $\mathbf{N}$ is not bounded we argue as follows. Assume it is bounded and let α be the least upper bound for $\mathbf{N}$. We would now have that for all $n \in \mathbf{N}$ that $n \leq \alpha$. Since $n+1$ is also a natural number we would have that $n + 1 \leq \alpha$ as well. So, in fact, $n \leq \alpha - 1$ or in other words, since n was any natural number, $\alpha - 1$ would be an upper bound contradicting that α was the least upper bound. □

As a corollary of the Archimedean property we get the following:

Corollary: If $x < y$, then there exists a rational number $\frac{m}{n}$ such that

$$x < \frac{m}{n} < y\,.$$

Proof. Set $\beta = \frac{1}{y-x}$. From the Archimedean property we have that there exists a natural number n with $n > \beta$. It follows that

$$0 < \frac{1}{n} < \frac{1}{\beta}\,.$$

Now let $m-1$ be the largest integer so that

$$m - 1 \leq x\, n\,.$$

It follows that $\frac{m}{n}$ has the desired property. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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