SPRING 2025 - 18.100B/18.1002

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Lecture 3

Theorem: $\mathbf{R}$ is a complete ordered Field that contains $\mathbf{Q}$.

$\mathbf{S}$ is an ordered set. A non-empty subset A of $\mathbf{S}$ is said to have an upper bound if there exists an $M \in \mathbf{S}$ such that for all $a \in A$ we have that $a \leq M$.

Completeness is the property that every bounded non-empty subset has a least upper bound.

We denote by $\sup A$ the smallest upper bound of A.

Lower bound: A non-empty subset A is said to have a lower bound if there exists $m \in \mathbf{S}$ such that for all $a \in A$ we have that $m \leq a$.

The greatest lower bound is a lower bound that is greater or equal to all other lower bounds.

The greatest lower bound is denoted by $\inf A$.

From now on we will concentrate of the case of $\mathbf{R}$.

How to write a mathematical proof?

This lecture we will look at how to write a mathematical proof.
We will explain this in two results that we talked about last time.

Let us return to the example of showing that $\sqrt{2} \in \mathbf{R}$.

We already showed this but we did not write it as a "proper proof". That is what we will do next.

Theorem: There exists $\alpha > 0$ such that $\alpha^2 = 2$.

Proof. Define a set A by

$$A = \{x \in \mathbf{R} \,|\, x > 0 \text{ and } x^2 \leq 2\}\,.$$

We will show that A is a non-empty bounded subset and that $\alpha = \sup A$ has the property that $\alpha > 0$ and $\alpha^2 = 2$.

Observe first that $1 \in A$, so A is non-empty. Moreover, 2 is an upper bound for A so A is bounded from above. Let $\alpha = \sup A$, we need to show that $\alpha > 0$ and that $\alpha^2 = 2$. Since $1 \in A$ it follows that $0 < 1 < \alpha$. To show that $\alpha^2 = 2$ we divide the proof into two parts.

Part 1: We will show that $\alpha^2 \leq 2$. Suppose not; we will see that this lead to a contradiction. Indeed, we will show that that if this was the case, then there exists an $0 < \alpha_0 < \alpha$ such that $\alpha_0^2 > 2$ so α_0 is an upper bound that is smaller than α. To show this we set

$$h = \frac{\alpha^2 - 2}{4\,\alpha}$$

and set

$$\alpha_0 = \alpha - h\,.$$

Note that since we are assuming that $\alpha > 2$, then we have that $h > 0$ and therefore $\alpha_0 < \alpha$. Note also that since $1 \leq \alpha \leq 2$ we have that

$$h \leq \frac{1}{2\,\alpha} \leq \frac{1}{2}\,.$$

In particular, $0 < \alpha_0$. Next

$$\alpha_0^2 = \alpha^2 + h^2 - 2\,h\,\alpha > \alpha^2 - \frac{\alpha^2 - 2}{2} = \frac{\alpha^2}{2} + 1 \geq 2\,.$$

This is the desired contradiction and show that $\alpha^2 \leq 2$.

Part 2: We will next show that $\alpha^2 \geq 2$. Suppose not; we will see that this lead to a contradiction. Indeed, we will show that if this was the case, then there exists an $\alpha_1 > \alpha$ such that $\alpha_1^2 < 2$ contradicting that α was an upper bound for A. So assume that $\alpha^2 < 2$. This time we will set

$$h = \frac{2 - \alpha^2}{4\,\alpha}\,.$$

Note that $1 > h > 0$ (the first inequality follows from that $1 \leq \alpha$). Set $\alpha_1 = \alpha + h$. It follows that

$$\alpha_1^2 = \alpha^2 + h^2 + 2\,h\,\alpha < \alpha^2 + h + \frac{2 - \alpha^2}{2} \leq \alpha^2 + 2\,\frac{2 - \alpha^2}{2} = 2\,.$$

Together parts 1 and 2 show that $\alpha^2 = 2$; completing the proof. $\square$

Archimedean property:

Formal proof:

Theorem: The set of natural number is not bounded from above.

Proof. If $\mathbf{N}$ is bounded from above, then we can let M be the least upper bound. We now have that for all $n \in \mathbf{N}$

$$n \leq \alpha$$

We claim that also $\alpha - 1$ is an upper bound contradicting that α was the least upper bound. Namely, for a given n since α is an upper bound for all natural numbers we have that

$$n + 1 \leq \alpha$$

but this implies that

$$n \leq \alpha - 1$$

showing that $\alpha - 1$ is an upper bound. That is the desired contradiction. □

Corollary: For any $\epsilon > 0$, there exists an $n \in \mathbf{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. Set $\alpha = \frac{1}{\epsilon}$, By the Archimedean property we know that there exists an $n \in \mathbf{N}$ with $n > \alpha$. It follows that $\frac{1}{n} < \epsilon$. □

Sequences:

$\sqrt{2}$ can be thought of a limit of a sequence of decimal numbers as follows.

$$1 < 1.4 < 1.41 < 1.414 \cdots .$$

When does a limit exist?

A sequence of real numbers is a function $f : \mathbf{N} \to \mathbf{R}$.

We usually use the notation $a_n = f(n)$.

Example 1: $\sqrt{2}$ is the limit of $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, $a_4 = 1.414$ etc.

Example 2: $a_n = (-1)^n$. This sequence has NO limit. The a_n's alternates between -1 and 1.

Example 3: The sequence $a_n = \frac{1}{n}$ has zero as its limit.

Limit: Let a_n be a sequence and a a real number. We say that a_n converges to a if for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that if $n \geq N$, then

$$|a_n - a| < \epsilon .$$

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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