SPRING 2025 - 18.100B/18.1002

TOBIAS HOLCK COLDING

Lecture 4

Theorem: $\mathbf{R}$ is a complete ordered Field that contains $\mathbf{Q}$.

$\mathbf{S}$ is an ordered set. A non-empty subset A of $\mathbf{S}$ is said to have an upper bound if there exists an $M \in \mathbf{S}$ such that for all $a \in A$ we have that $a \leq M$.

Completeness is the property that every bounded non-empty subset has a least upper bound.

We denote by $\sup A$ the greatest lower bound of A.

Lower bound: A non-empty subset A is said to have a lower bound if there exists $m \in \mathbf{S}$ such that for all $a \in A$ we have that $m \leq a$.

The greatest lower bound is a lower bound that is greater or equal to all other lower bounds.

Sequences:

$\sqrt{2}$ can be thought of a limit of a sequence of decimal numbers as follows.

$$1 < 1.4 < 1.41 < 1.414 \cdots .$$

A sequence of real numbers is a function $f : \mathbf{N} \to \mathbf{R}$.

We usually use the notation $a_n = f(n)$.

Example 1: $\sqrt{2}$ is the limit of $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, $a_4 = 1.414$ etc.

Example 2: $a_n = (-1)^n$. This sequence has NO limit. The a_n's alternates between -1 and 1.

Example 3: The sequence $a_n = \frac{1}{n}$ has zero as its limit.

Limit: Let a_n be a sequence and a a real number. We say that a_n converges to a if for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that if $n \geq N$, then

$$|a_n - a| < \epsilon\,.$$

If this is the case, then we also say that a is the limit of the sequence and we say that the sequence is convergenet.

A sequence that is not covergent is said to be divergent.

Example:

$$0.999999999\cdots = 1\,.$$

What does the left hand side mean?

Define a sequence a_n as follows: Set

$$a_1 = 0.9\,,$$

$$a_2 = 0.99\,,$$

$$a_3 = 0.999\,,$$

$$a_4 = 0.9999\,,$$

etc.

The left hand side above is then defined as the limit of the sequence a_n.

Claim:

$$\lim_{n\to\infty} a_n = 1\,.$$

Proof. We need to show that for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that for $n \geq N$ we have that

$$|a_n - 1| < \epsilon\,.$$

By the Archimedean property we can choose N such that $\frac{1}{N} < \epsilon$. We also have that

$$|a_n - 1| = 10^{-n}\,.$$

Therefore, for $n \geq N$ we have that

$$|a_n - 1| = 10^{-n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon\,.$$

This proves the claim. □

Theorem If a_n is a convergent sequence, then the set $\{a_n\}$ is a bounded subset of $\mathbf{R}$.

Proof. Since a_n is convergent to a we can find N such that for $n \geq N$ we have that

$$|a - a_n| < 1\,.$$

Note also that the set $\{a_1, \cdots, a_{N-1}\}$ is bounded so there exists $C \in \mathbf{R}$ such that for $n = 1, \cdots, N-1$ we have that

$$|a_n| \leq C\,.$$

To see that the larger set $\{a_n\}$ is bounded we will use that for $n \geq N$

$$|a_n| \leq |a| + |a_n - a| \leq |a| + 1\,.$$

From this we have that for all n

$$|a_n| \leq \max\{C, |a| + 1\}\,.$$

□

Basic algebraic properties of limits:

Theorem Suppose that a_n and b_n are convergenet sequences with $\lim a_n = a$, $\lim b_n = b$ and $C \in \mathbf{R}$, then

1. $c_n = C\,a_n$ is convergenet and $\lim_{n\to\infty} c_n = C\,a$.
2. $c_n = a_n + b_n$ is convergent and $\lim_{n\to\infty} c_n = a + b$.
3. $c_n = a_n\,b_n$ is convergent with $\lim_{n\to\infty} c_n = a\,b$.
4. If $b_n \neq 0$, $b \neq 0$ and $c_n = \frac{a_n}{b_n}$, then c_n is convergent and $\lim_{n\to\infty} c_n = \frac{a}{b}$.

Proof. (of the first property.) If $c = 0$, then the claim is obviously true so we need only show the claim for $C \neq 0$. Given $\epsilon > 0$, there exists an N such that if $n \geq N$, then

$$|a - a_n| < \frac{\epsilon}{|C|}\,.$$

Multiplying both sides by $|C|$ gives that

$$|C\,a - C\,a_n| < \epsilon\,.$$

for $n \geq N$. This show the first property.

(Of the second of these properties.) Observe that

$$|c_n - (a+b)| = |a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|\,.$$

Since $a_n \to a$, given $\epsilon > 0$ we can find a N_a such that if $n \geq N_a$, then

$$|a_n - a| < \frac{\epsilon}{2}\,.$$

Likewise since $b_n \to b$ we can find N_b such that if $n \geq N_b$, then

$$|b_n - b| < \frac{\epsilon}{2}\,.$$

We now set $N = \max\{Na, N_b\}$ and observe that if $n \geq N$, then

$$|c_n - (a+b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

This proves the second property.

(Outline of how to show the third property.) To prove the third property we will use that

$$|a\,b - a_n\,b_n| \leq |a\,b - a_n\,b| + |a_n\,b - a_n\,b_n| = |b|\,|a - a_n| + |a_n|\,|b - b_n|\,.$$

We then combine it with the theorem above that show that the set

$$\{|a_n|\,|\,n \in \mathbf{N}\}$$

is bounded. This is the main idea of the proof of the third property. There are the details to fill in to make it a proof.

(Outline of how to show the fourth property.) To prove the fourth property we will assume that $a_n = 1$. The general case indeed will follow from this together with the third property. We will use that

$$\frac{1}{b_n} - \frac{1}{b} \;= \frac{|b - b_n|}{|b|\,|b_n|}\,,$$

together with that

$$|b_n| \leq |b| + |b_n - b|\,.$$

and therefore

$$|b_n| \geq |b| - |b_n - b|\,.$$

We then want to use this to bound the denominator (when n is sufficiently large) from below in

$$\frac{|b - b_n|}{|b|\,|b_n|}\,.$$

Like for the third property there are details to fill in but these are the main ideas. □

Subsequence:

Example 1: Suppose that $a_n = (-1)^n$.
This is a sequence of 1's and -1's that is alternating between -1 and 1.
The sequence $b_n = 1$ for all n is a subsequence.
Another subsequence is where $c_n = -1$.
Also the sequence $c_n = (-1)^{n+1}$ is a subsequence of a_n.
Another example of a subsequence is

$$1, 1, -1, -1, 1, 1, -1, -1, \cdots .$$

Example 2: Suppose $a_n = n$. So a_n is:

$$1, 2, 3, 4, 5, 6, \cdots .$$

The sequence of increasing odd numbers

$$1, 3, 5, 7, 9, \cdots .$$

is a subsequence.
The sequence of increasing even numbers

$$2, 4, 6, 8, 10, \cdots .$$

is another subsequence.
The sequence

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5 \cdots ,$$

is NOT s subsequence.

Formel definition: Recall that a sequence a_n is a function $f : \mathbf{N} \to \mathbf{R}$ where we set $a_n = f(n)$. A subsequence b_n of a_n is a composition of functions $f \circ g$ where $g : \mathbf{N} \to \mathbf{N}$ is a strictly increasing function. So $b_n = f(g(n))$. Sometimes a subsequence of the sequence a_n also denoted by a_{n_k}.

Theorem: A sequence a_n is convergent with limit a if and only if all subsequences of a_n are also convergent with limit a.

Proof. We need to show two implications.

First we need to show that is all subsequences of a_n are convergent with limit a, then the sequence a_n is convergent with limit a. However, this is trivially so since a_n itself is a (trivial) subsequence of a_n.

Next we need to show that any subsequence of a convergent sequence is convergent with the same limit. Suppose therefore that $\epsilon > 0$ is given and choose N so large so that for

$n \geq N$

$$|a_n - a| < \epsilon\,.$$

For $k \geq N$ we have that $n_k \geq k \geq N$ and therefore

$$|a_{n_k} - a| < \epsilon\,.$$

This proves the second implication. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
(screen-optimized)
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Portrait.pdf
(print-optimized)

MIT, Dept. of Math., 77 Massachusetts Avenue, Cambridge, MA 02139-4307.

18.100B Real Analysis
Spring 2025