SPRING 2025 - 18.100B/18.1002

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Lecture 5

A sequence of real numbers is a function $f : \mathbf{N} \to \mathbf{R}$.

We usually use the notation $a_n = f(n)$.

Limit: Let a_n be a sequence and a a real number. We say that a_n converges to a if for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that if $n \geq N$, then

$$|a_n - a| < \epsilon\,.$$

If this is the case, then we also say that a is the limit of the sequence and we say that the sequence is convergenet. A sequence that is not covergent is said to be divergent.

Theorem If a_n is a convergent sequence, then the set $\{a_n\}$ is a bounded subset of $\mathbf{R}$.

Basic algebraic properties of limits:

Theorem Suppose that a_n and b_n are convergenet sequences with $\lim a_n = a$, $\lim b_n = b$ and $C \in \mathbf{R}$, then

(1) $c_n = C\, a_n$ is convergenet and $\lim_{n\to\infty} c_n = C\, a$.
(2) $c_n = a_n + b_n$ is convergent and $\lim_{n\to\infty} c_n = a + b$.
(3) $c_n = a_n\, b_n$ is convergent with $\lim_{n\to\infty} c_n = a\, b$.
(4) If $b_n \neq 0$, $b \neq 0$ and $c_n = \frac{1}{b_n}$, then c_n is convergent and $\lim_{n\to\infty} c_n = \frac{1}{b}$.

Subsequence: A subsequence b_k of a_n is a composition of functions $f \circ g$ where $g : \mathbf{N} \to \mathbf{N}$ is a strictly increasing function. So $b_k = f(g(k))$. We often write a_{n_k} for b_k.

Example: A sequence is defined by a_n

$$a_n = \frac{n^2+1}{n^2+n+1}\,.$$

We will show that a_n is convergent with limit 1. We have

$$a_n = \frac{1+\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}}\,.$$

Since $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$ we have that

$$1 + \frac{1}{n^2} \to 1$$

and

$$1 + \frac{1}{n} + \frac{1}{n^2} \to 1\,.$$

It now follows from the algebraic properties of limits that

$$a_n \to 1\,.$$

Example: To show that $\sqrt{2}$ is the limit of $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, $a_4 = 1.414$, ... we need something else. We need the monotone convergence theorem.

Monotone convergence theorem: Increasing version. Let a_n be a monotone increasing sequence. This means that $a_1 \leq a_2 \leq a_3 \leq \cdots$. (Which we can also write this as $a_n \leq a_{n+1}$). If the sequence is bounded from above so that there exists A with

$$a_n \leq A\,,$$

then a_n is convergent with limit $\sup\{a_n\}$.

Monotone convergence theorem: Decreasing version. Similarly, for a bounded monotone decreasing sequence a_n where $a_{n+1} \leq a_n$, we have that a_n converges and

$$\lim_{n\to\infty} a_n = \inf\{a_n\}\,.$$

Using the monotone convergence theorem we can now show that $\sqrt{2}$ is the limit of $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, $a_4 = 1.414$, $a_5 = \cdots$.

To show this we argue as follows:

Let
$$a_n = \frac{b_n}{10^{n-1}}$$
where b_n is the largest integer so that
$$b_n^2 \leq 2 \cdot 10^{2\,n-2}\,.$$

We will show that a_n is an increasing and bounded sequence and that the limit a has the property that $a^2 = 2$.

So suppose that b_n is an integer and that
$$b_n^2 \leq 2 \cdot 10^{2\,n-2}\,,$$
then obviously
$$(10\,b_n)^2 \leq 2 \cdot 10^{2\,(n+1)-2}\,.$$

This implies that $b_{n+1} \geq 10\,b_n$; so $a_{n+1} \geq a_n$ and the sequence is increasing.

It is clear that $b_n \leq 2 \cdot 10^{n-1}$, since $(2 \cdot\; 10^{2\,n-2})^2 > 2 \cdot 10^{2\,n-2}$.

Therefore the sequence $a_n = \frac{b_n}{10^{n-1}}$ is bounded and by the monotone convergence theorem $a_n \to a$.

To show that $a^2 = 2$ we argue as follows:

By the algebraic properties of limits we have that $a^2 = \lim_{n\to\infty} a_n^2$, but for each n we have that $a_n^2\, 10^{2\,n-2} = b_n^2 \leq 2 \cdot 10^{2\,n-2}$ so $a_n^2 \leq 2$.

This show that $a^2 \leq 2$.

Similarly, since $(b_n + 1)^2 > 2 \cdot 10^{2\,n-2}$ we have that $(a_n + 10^{1-n})^2 = \frac{(b_n+1)^2}{10^{2\,n-2}} > 2$.

Therefore, $2 \leq \lim_{n\to\infty}(a_n + 10^{1-n})^2 = \lim_{n\to\infty} a_n^2 = a^2$.

Proof of the monotone convergence theorem (Increasing version):

The decreasing version is proven similarly.

So suppose that we have a sequence with

$$a_n \leq a_{n+1} \leq A$$

and set

$$a = \sup \{a_n\} .$$

We want to show that

$$a_n \to a .$$

Given $\epsilon > 0$, since $a - \epsilon < a$ we have that $a - \epsilon$ is not an upper bound for the sequence, therefore there exists N such that

$$a_N > a - \epsilon .$$

Since the sequence is increasing we have for $n \geq N$ that

$$a - \epsilon < a_N \leq a_n \leq a$$

Here the last inequality used that a is an upper bound for the sequence.

We now have that for $n \geq N$

$$0 \leq a - a_n < \epsilon .$$

This shows that the sequence is convergent with limit a.

Cauchy sequence: A sequence a_n is said to be a Cauchy sequence if for all $\epsilon > 0$, there exists an N such that if m, $n \geq N$, then

$$|a_n - a_m| < \epsilon .$$

(Tail of the sequence bunch together.)

From Wikipedia: Baron Augustin-Louis Cauchy (1789 – 1857) was a French mathematician, engineer, and physicist. He was one of the first to rigorously state and prove the key theorems of calculus (thereby creating real analysis), pioneered the field complex analysis, and the study of permutation groups in abstract algebra. Cauchy also contributed to a number of topics in mathematical physics, notably continuum mechanics.

Theorem: A sequence is convergent if and only if it is a Cauchy sequence.

Application: Existence of fixed points for a maps. If $T : \mathbf{R} \to \mathbf{R}$ is a map, then $x_0 \in \mathbf{R}$ is a fixed point if

$$T(x_0) = x_0 .$$

Definition A contracting map is a map $T : \mathbf{R} \to \mathbf{R}$ such that there exists $c < 1$ so for all x, $y \in \mathbf{R}$ we have that

$$|T(x) - T(y)| \leq c\,|x - y| .$$

(Points are squeezed together under the map.)

Contracting mapping theorem: Any contracting map has a fixed point.

Application of contracting mapping theorem: Existence of solutions to ODEs.

More on all of this next time......

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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