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Lecture 6

Last time:

Basic algebraic properties of limits.

Monotone convergence theorem.

Cauchy sequence.

Cauchy sequence: A sequence a_n is said to be a Cauchy sequence if for all $\epsilon > 0$, there exists an N such that if m, $n \geq N$, then

$$|a_n - a_m| < \epsilon\,.$$

(Tail of the sequence bunch together.)

Theorem (Cauchy convergence theorem): A sequence is convergent if and only if it is a Cauchy sequence.

Application: Existence of fixed points for a maps.
If $T : \mathbf{R} \to \mathbf{R}$ is a map, then $x_0 \in \mathbf{R}$ is a fixed point if

$$T(x_0) = x_0\,.$$

Definition A contracting map is a map $T : \mathbf{R} \to \mathbf{R}$ such that there exists $c < 1$ so for all x, $y \in \mathbf{R}$ we have that

$$|T(x) - T(y)| \leq c\,|x - y|\,.$$

(Points are squeezed together under the map.)

Contracting mapping theorem: Any contracting map has a fixed point.

For a contracting map the fix point is unique.

Suppose that x and y are two fixed point we want to show that $x = y$. We have

$$|x - y| = |T(x) - T(y)| \leq c\,|x - y|\,.$$

Since $c < 1$ this implies that $|x - y| = 0$ and so $x = y$.

On Pset 3 you will be asked to show that for a contracting map T and any $a_1 \in \mathbf{R}$ the sequence $a_{n+1} = T(a_n)$ is a Cauchy sequence. By the Cauchy theorem we then have that a_n is convergent.

Let a denote the limit. We claim that $T(a) = a$. Observe that $T(a_n) = a_{n+1} \to a$. If we can show that if $x_n \to x$, then $T(x_n) \to T(x)$, then

$$T(a_n) \to T(a)$$

but we already have that $T(a_n) = a_{n+1} \to a$ so we would have that $T(a) = a$ and thus a is a fixed point.

We need therefore show that if $x_n \to x$, then $T(x_n) \to T(x)$. To do that observe that

$$|T(x_n) - T(x)| \leq c\,|x_n - x|\,.$$

Since $x_n \to x$ we have that $|x_n - x| \to 0$ and so $|T(x_n) - T(x)| \to 0$. It follows that $T(x_n) \to T(x)$. Applying this to the sequence a_n shows that a is a fixed point for T.

Applications of contracting mapping theorem:

Existence of solutions to ODEs. We will return to this later as this needs a version of the contracting mapping theorem where T is defined on a more general space than the real numbers.

Newton's method: Finding a zeroth of a function $f : \mathbf{R} \to \mathbf{R}$. (So find a solution x to $f(x) = 0$.)

Suppose that x_1 is a "good" initial guess, so $f(x_1)$ is sufficiently small. Assume also that $f' \neq 0$. Define a map

$$T(x) = x - \frac{f(x)}{f'(x)}\,.$$

We have

$$T'(x) = 1 - \frac{f'}{f'} + \frac{f\,f''}{(f')^2} = f\,\frac{f''}{(f')^2}\,.$$

So as long as x stay close to the initial guess and for the initial guess $f(x)$ is small compared with $\frac{f''}{(f')^2}$, then T is a contracting map. By the contracting mapping theorem the sequence $x_{n+1} = T(x_n)$ is a Cauchy sequence that converges to a fixed point of T.

For a fixed point for T we have $T(x) = x$ so $x - \frac{f(x)}{f'(x)} = x$ and therefore $f(x) = 0$.

Back to Cauchy sequences.

Bolzano -Weirstrass theorem: Any bounded sequence has a convergent subsequence.

Once we have the Bolzano-Weirstrass theorem we can prove the Cauchy theorem.

Proof. (of the Cauchy theorem.) So suppose that a_n is a Cauchy sequence. We will first show that a_n is bounded. From the definition of a Cauchy sequence we have that there exists N such that for m, $n \geq N$, then

$$|a_n - a_m| < 1\,.$$

It follows, in particular, that for all $n \geq N$, we have that

$$|a_n - a_N| < 1\,,$$

and so

$$|a_n| = |(a_n - a_N) + a_N| \leq 1 + |a_N|\,.$$

Therefore,

$$|a_n| \leq \max\{|a_N| + 1, |a_1|, \cdots, |a_{N-1}|\}\,.$$

So the sequence is bounded.

From the Bolzano-Weirstrass theorem it follows that a_n has a convergent subsequence a_{n_k} with limit a. We want to show that a_n is convergent with limit a. Given $\epsilon > 0$, there exists an N_1 such that if m, $n \geq N_1$, then

$$|a_n - a_m| < \frac{\epsilon}{2}\,.$$

Moreover, there exist an N_2 such that if $k \geq N_2$, then $|a_{n_k} - a| < \frac{\epsilon}{2}$. Set $N = \max\{N_1, N_2\}$. It follows that if $n \geq N$ and $k \geq N$, then

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2}\,.$$

This show that $a_n \to a$ as claimed. □

Another application of the Bolzano-Weirstrass theorem is the Extreme value theorem.

Before stating this we need another key notion:

A function $f : \mathbf{R} \to \mathbf{R}$ is said to be continuous at a point $x_0 \in \mathbf{R}$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon\,.$$

A function is said to be continuous if it is continuous at all points in the domain.

Theorem: If $f : \mathbf{R} \to \mathbf{R}$ is continuous and x_n is a sequence with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Proof. Given $\epsilon > 0$, since f is continuous, there exists a $\delta > 0$, such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. Since $x_n \to x_0$, there exists N such that if $n \geq N$, then $|x_n - x_0| < \delta$ and therefore $|f(x_n) - f(x_0)| < \epsilon$. This show that $f(x_n) \to f(x_0)$ as claimed. □

Extreme value theorem: Let f be a continues function on an interval $[a, b]$. The extreme value theorem says that the sup and inf are achieved. That is, there exist $x \in [a, b]$ such that $f(x) = \sup f$. Likewise for $\inf f$.

Proof. We will show that the supremum is achieved. The proof that the infimum is the same with obvious modification. Let $x_n \in [a, b]$ be a sequence where $f(x_n) \to \sup f$. Since the sequence is contained in $[a, b]$ it is bounded and therefore by the Bolzano - Weirstrass theorem has a convergent subsequence $x_{n_k} \to x$. Note that $x \in [a, b]$. By the theorem above $f(x_{n_k}) \to f(x)$ and since we also have that $f(x_{n_k}) \to \sup f$ it follows that $\sup f = f(x)$. This proves that the supremum is achieved. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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