SPRING 2025 - 18.100B/18.1002

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Lecture 7

Last time:

Cauchy convergence theorem: Every Cauchy sequence is convergent.

Bolzano-Weirstrass theorem.

Last time we showed that the Bolzano-Weirstrass implies the Cauchy theorem.

Bolzano-Weirstrass theorem: Any bounded sequence has a convergent subsequence.

Proof. (of the Bolzano-Weirstrass theorem.) Suppose that a_n is a bounded sequence. For simplicity assume that $a_n \in [0, 1]$.

Defining the subsequence a_{n_k}. Either there are infinite many n such that $a_n \in [0, \frac{1}{2}]$ or there are infinite many such a_n in $[\frac{1}{2}, 1]$ (or both). Assume that there are infinitely many in $[0, \frac{1}{2}]$. Set $a_{n_1} = a_1$. Let a_{n_2} be the next a_n such that $a_n \in [0, \frac{1}{2}]$. We have

$$n_2 > n_1 = 1\,,$$

$$a_{n_1} \in [0, 1]\,,$$

$$a_{n_2} \in \left[0, \frac{1}{2}\right]\,.$$

Next either infinitely many a_n lies in $[0, \frac{1}{4}]$ or infinitely many a_n lies in $[\frac{1}{4}, \frac{1}{2}]$. Assume that infinitely many lies in $[\frac{1}{2}, \frac{1}{4}]$. Pick an $n > n_2$ such that $a_n \in [\frac{1}{4}, \frac{1}{2}]$ and set $a_{n_3} = a_n$. We continue this way.

Convergence of a_{n_k}. Note that for $k_1, k_2 \geq k$ we have that

$$|a_{n_{k_1}} - a_{n_{k_2}}| \leq 2^{1-k}\,.$$

Since $2^{-k} \to 0$ as $k \to \infty$ this shows that the subsequence a_{n_k} is a Cauchy sequence. However, more is true. Squeezed between two other sequences $b_k \leq a_{n_k} \leq c_k$. We will define sequences b_k and c_k as follows. The sequence b_k will be the left endpoint of the interval of length 2^{1-k} that all the element a_{n_i} will lie in when $i \geq k$ and c_k will be the right end point of the same interval. We have now that the sequence b_k is increasing and the sequence c_k is decreasing and the a_{n_k} are squeezed between the two. It follows that b_k is convergent (as it is also bounded) and likewise for c_k. Since $c_k - b_k = 2^{1-k}$ it follows that b_k and c_k converges to the same number and since the later a_{n_k} are all squeezed between the two they also converges to that same number. □

From Wikipedia: Karl Theodor Wilhelm Weierstrass (1815 – 1897) was a German mathematician often cited as the "father of modern analysis". Despite leaving university without a degree, he studied mathematics and trained as a school teacher, eventually teaching mathematics, physics, botany and gymnastics. Among many other contributions, Weierstrass formalized the definition of the continuity of a function and complex analysis, proved the intermediate value theorem and the Bolzano–Weierstrass theorem, and used the latter to study the properties of continuous functions on closed bounded intervals.

Series: Suppose that a_n is a sequence, we can form a new sequence s_n as follows. We let

$$s_1 = a_1 \,,$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3 \,,$$

and in general set

$$s_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i \,.$$

A series $\sum_{i=1}^{\infty} a_i$ converges if the sequence s_n converges and if it do we also write $\sum_{i=1}^{\infty} a_i$ for the limit.

Geometric series: Suppose now that $a_n = c^n$ so the series is

$$s_n = \sum_{i=0}^{n} c^i \,.$$

This is the geometric series. It is convergent precisely when $|c| < 1$. Moreover, when $|c| < 1$, then the limit (infinite sum) is

$$\sum_{i=0}^{\infty} c^i = \frac{1}{1-c} \,.$$

To see this observe that

$$(1-c) \sum_{i=0}^{n} c^i = 1 - c^{n+1} \,.$$

It follows from this that if $c \neq 1$, then

$$s_n = \sum_{i=0}^{n} c^i = \frac{1 - c^{n+1}}{1-c} \,.$$

Therefore, s_n converges with limit $\frac{1}{1-c}$ if $|c| < 1$ and diverges if $|c| > 1$ or $c = -1$. One easily checks, separately, that it also diverges for $c = 1$.

Harmonic series: For the harmonic series $a_n = \frac{1}{n}$ so the series is $\sum_{i=1}^{\infty} \frac{1}{2}$. This series is divergent. To see this we will show that

$$s_{2^n-1} \geq \frac{n}{2}\,.$$

This is true for $n = 1$ as $s_1 = 1 \geq \frac{1}{2}$.

Assume that it is true for n we will show that it is also true for $n+1$. Namely,

$$s_{2^{n+1}-1} \geq s_{2^n-1} + \sum_{i=2^n}^{2^{n+1}-1} \frac{1}{i} \geq \frac{n}{2} + 2^n \frac{1}{2^{n+1}-1} \geq \frac{n}{2} + \frac{1}{2} = \frac{n+1}{2}\,.$$

So the formula also holds for $n+1$ and therefore for all n.

Since $\frac{n}{2} \to \infty$ it follows that the subsequence s_{2^n-1} is divergent and therefore so is the original series.

Absolutely convergent; We say that a series

$$\sum_{n=0}^{\infty} a_n$$

is absolutely convergent if the series

$$\sum_{n=0}^{\infty} |a_n|$$

is convergent. Absolutely convergent implies convergent but not the other way around.

Example: We will see later that the series

$$\sum_{n=1} \frac{(-1)^n}{n}$$

is convergent but if we take the absolute values of the a_n's, then we get the harmonic series which is divergent.

Proof. (of why absolutely convergent implies convergent.) By the Cauchy convergence theorem we only need to show that the sequence

$$\sum_{i=0}^{\infty} a_i$$

is Cauchy sequence if the series

$$\sum_{n=0}^{\infty} |a_n|$$

is convergent. Set

$$s_n = \sum_{i=0}^{n} a_i \, ,$$

$$\mathbf{s}_n = \sum_{i=0}^{n} |a_i| \, .$$

For $m < n$ we have

$$|s_n - s_m| = |a_n + \cdots + a_{m+1}| \leq |a_n| + \cdots + |a_{m+1}| = |\mathbf{s}_n - \mathbf{s}_m| \, .$$

Since the sequence $\mathbf{s}_n$ is a Cauchy sequence it now follows that s_n is. □

Theorem: A series of non-negative numbers

$$\sum_{i=0}^{\infty} a_i \, ,$$

where $a_n \geq 0$, is convergent if and only if the sequence s_n is bounded from above.

Proof. The sequence s_n is monotone nondecreasing since

$$s_{n+1} = s_n + a_n \geq s_n \, .$$

The claim now follows from the monotone convergence theorem. □

Example: The series

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

is convergent. This is a sequence of non-negative numbers so we only need to show that there exist M such that for all n

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2} \leq M \, .$$

Claim:

$$s_{2^n - 1} \leq \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i \, .$$

This would be enough because the last is a convergent geometric series so in particular bounded.

We will show this by induction. For $n = 1$ we have that

$$s_1 = 1 = \left(\frac{1}{2}\right)^0 .$$

So it is correct for $n = 1$. Assume next that it is true for n; we will show that it also holds for $n + 1$.

$$s_{2^{n+1}-1} = s_{2^n-1} + \sum_{i=2^n}^{2^{n+1}-1} \frac{1}{i^2} \leq \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i + 2^n \frac{1}{(2^n)^2}$$
$$\leq \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i + \frac{1}{2^n} = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i .$$

This show the induction step and completes the proof.

To help determine whether or not a series converges there are a number of tests:

- Comparison test.
- Ratio test.
- Root test.

Comparison test; version 1: Suppose that a_n and b_n are two sequences with

$$0 \leq a_n \leq b_n .$$

If

$$\sum_{n=1}^{\infty} b_n$$

is convergent, then so is

$$\sum_{n=1}^{\infty} a_n .$$

Example: The series

$$\sum_{n=1}^{\infty} \frac{2^{-n}}{n}$$

is convergent. Namely, if we set

$$a_n = \frac{2^{-n}}{n}$$

and

$$b_n = 2^{-n} ,$$

then $0 \leq a_n \leq b_n$ and since the series $\sum_{n=1}^{\infty} b_n$ is convergent, then by the comparison test so is the series $\sum_{n=1}^{\infty} a_n$.

Comparison test; version 2: Suppose that a_n and b_n are two sequences with $b_n \neq 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0\,,$$

The series

$$\sum_{n=1}^{\infty} a_n$$

is convergent if and only if

$$\sum_{n=1}^{\infty} b_n$$

is.

Example: The series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

is convergent since

$$\frac{n^2}{n^2-1} \to 1\,,$$

and the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is convergent.

Ratio test: Let $a_n \geq 0$ and assume that

$$\frac{a_{n+1}}{a_n} \to a$$

.

If

- $a < 1$, then the series $\sum a_n$ is convergent.
- $a > 1$, then the series $\sum a_n$ is divergent.
- $a = 1$, it is inconclusive.

Example 1:

$$a_n = \frac{1}{n}\,.$$

In this case

$$\frac{a_{n+1}}{a_n} \to 1$$

so the test is inconclusive, but the series is divergent.

Example 2:

$$a_n = \frac{1}{n^2}\,.$$

In this case

$$\frac{a_{n+1}}{a_n} \to 1$$

so the test is inconclusive, but the series is convergent.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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18.100B Real Analysis
Spring 2025