SPRING 2025 - 18.100B/18.1002

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Lecture 8

Series: Suppose that a_n is a sequence, we can form a new sequence s_n as follows. We let

$$s_1 = a_1 \,,$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3 \,,$$

and in general set

$$s_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i \,.$$

A series $\sum_{i=1}^{\infty} a_i$ converges if the sequence s_n converges and if it do we also write $\sum_{i=1}^{\infty} a_i$ for the limit.

Geometric series:

$$\sum_{i=0}^{\infty} c^i n \,.$$

Convergent precisely when $|c| < 1$.

Harmonic series:

$$\sum_{i=1}^{\infty} \frac{1}{n} \,.$$

This series is divergent.

Absolutely convergent; We say that a series

$$\sum_{n=0}^{\infty} a_n$$

is absolutely convergent if the series

$$\sum_{n=0}^{\infty} |a_n|$$

is convergent. Absolutely convergent implies convergent but not the other way around.

Theorem: A series of non-negative numbers $a_n \geq 0$

$$\sum_{i=0}^{\infty} a_i \,,$$

is convergent if and only if the sequence s_n is bounded from above.

To help determine whether or not a series converges there are a number of tests:
- Comparison test.
- Ratio test.
- Root test.
- Other tests that we will discuss later.

Comparison test; version 1: Suppose that a_n and b_n are two sequences with

$$0 \leq a_n \leq b_n \,.$$

If

$$\sum_{n=1}^{\infty} b_n$$

is convergent, then so is

$$\sum_{n=1}^{\infty} a_n \,.$$

Example: The series

$$\sum_{n=1}^{\infty} \frac{2^{-n}}{n}$$

is convergent. Namely, if we set

$$a_n = \frac{2^{-n}}{n}$$

and

$$b_n = 2^{-n} \,,$$

then $0 \leq a_n \leq b_n$ and since the series $\sum_{n=1}^{\infty} b_n$ is convergent, then by the comparison test so is the series $\sum_{n=1}^{\infty} a_n$.

Comparison test; version 2: Suppose that a_n and b_n are two sequences with $b_n \neq 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0\,,$$

The series

$$\sum_{n=1}^{\infty} a_n$$

is convergent if and only if

$$\sum_{n=1}^{\infty} b_n$$

is.

Example: The series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

is convergent since

$$\frac{n^2}{n^2-1} \to 1\,,$$

and the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is convergent.

Ratio test: Let $a_n \geq 0$ and assume that

$$\frac{a_{n+1}}{a_n} \to a$$

.

If

- $a < 1$, then the series $\sum a_n$ is convergent.
- $a > 1$, then the series $\sum a_n$ is divergent.
- $a = 1$, it is inconclusive.

Example 1:

$$a_n = \frac{1}{n}\,.$$

In this case

$$\frac{a_{n+1}}{a_n} \to 1$$

so the test is inconclusive, but the series is divergent.

Example 2:

$$a_n = \frac{1}{n^2}\,.$$

In this case

$$\frac{a_{n+1}}{a_n} \to 1$$

so the test is inconclusive, but the series is convergent.

Root test: Let $a_n \geq 0$ be a sequence of non-negative numbers. Suppose $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = r$. If

- $r < 1$, then the series $\sum_{n=0}^{\infty} a_n$ is convergent.
- $r > 1$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.
- $r = 1$, then it is inconclusive.

Proof. (of root test.) Suppose that $r < 1$. It follows that for $r < r_0 < 1$, there exists N such that if $n \geq N$, then

$$(a_n)^{\frac{1}{n}} \leq r_0\,.$$

Therefore,

$$0 \leq a_n \leq r_0^n\,.$$

However, the series $\sum_{n=0}^{\infty} r_0^n$ is a geometric series that is convergent since $r_0 < 1$. We now have by the first version of the comparison test that also the series $\sum_{n=0}^{\infty} a_n$ is convergent.

Suppose that $r > 1$. In that case we have that for $1 < r_0 < r$, there exists N such that if $n \geq N$, then

$$(a_n)^{\frac{1}{n}} \geq r_0\,.$$

Hence, for $n \geq N$

$$a_n \geq r_0^n\,,$$

where, the series $\sum_{n=0}^{\infty} r_0^n$ is a divergent geometric series. Therefore, by the comparison test the original series is divergent. □

Power series:

- $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$
- $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp\,x\,.$$

- $$\sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2\,n}}{(2\,n)!} = \cos\, x \, .$$
- $$\sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2\,n+1}}{(2\,n+1)!} = \sin\, x \, .$$

Formel definition: Let c_n be a sequence, then $\sum_{n=0}^{\infty} c_n \, x^n$ is a power series.

When does a power series converge?

Why does it give familiar functions?

We will answer the second question next time for the exponential function.

The answer to the first question comes from the root test or the ratio test.

Example: Consider the power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \, .$$

By the ratio test with $a_n = \frac{x^n}{n!}$ we have

$$\frac{a_{n+1}}{a_n} = \frac{n!\, x^{n+1}}{(n+1)!\, x^n} = \frac{x}{n+1} \to 0 \, .$$

It follows that the power series is convergent for all x.

Example:

$$\sum_{n=0}^{\infty} x^n \, .$$

This series is convergent for $|x| < 1$ and divergent otherwise.

To talk about convergence of a general power series we need the notion of lim sup of a sequence. This is defined as follows.

Let a_n be a sequence. If it is not bounded from above, then we set $\limsup a_n$ to be ∞. Otherwise we will define a new sequence b_n from a_n as follows.

$$b_n = \sup \{a_n, a_{n+1}, a_{n+2}, \cdots\} .$$

Note that since we are assuming that the a_n's are bounded from above the b_n's are real numbers and the sequence b_n is decreasing. – It is decreasing since

$$b_n = \sup \{a_n, a_{n+1}, a_{n+2}, \cdots\} \geq \sup \{a_{n+1}, a_{n+2}, \cdots\} = b_{n+1} .$$

(For b_{n+1} supremum is taken over a smaller set.)

Since the sequence b_n is decreasing it is converging with limit b that possibly could be $-\infty$ if the sequence b_n is not bounded from below.

Definition (of lim sup):

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n = b .$$

Back to power series. Suppose that

$$\sum_{n=0}^{\infty} a_n \, x^n$$

is a power series. Set

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} .$$

R is said to be the radius of convergence.

Convention: If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$, then the radius of convergence is said to be ∞. If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$, then we set $R = 0$.

From the root test one can now show the following:

The power series in convergent if $|x| < R$ and divergent if $|x| > R$.

The case of where $|x| = R$ has to be examined on a case by case basis.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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