SPRING 2025 - 18.100B/18.1002

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Lecture 9

Power series: Suppose that a_n is a sequence, for each x we can form a series

$$\sum_{n=0}^{\infty} a_n\, x^n\,.$$

Exponential map as a power series: Define $E(x)$ as the power series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.$$

Step 0: The power series converges for all x. Namely, since

$$\frac{|a_{n+1}\, x^{n+1}|}{|a_n\, x^n|} = \frac{|n!\, x^{n+1}|}{|(n+1)!\, x^n|} = \frac{x}{n+1} \to 0\,,$$

the claim follows from the ratio test.

Step 1: Define $\mathrm{e} = E(1)$ and $\mathrm{e}^2 = \mathrm{e}\,\mathrm{e}$, $\mathrm{e}^3 = \mathrm{e}\,\mathrm{e}\,\mathrm{e}$ etc. This way e^k is defined for all $k \in \mathbf{N}$.

We also define $\mathrm{e}^{-k} = \frac{1}{\mathrm{e}^k}$. (The idea is that we would like to have $\mathrm{e}^{x+y} = \mathrm{e}^x\,\mathrm{e}^y$.)

We set $\mathrm{e}^0 = 1$.

If $q = \frac{m}{n} \in \mathbf{Q}$, where $m \in \mathbf{Z}$ and $n \in \mathbf{N}$, then we let e^q be the positive number α so that $\alpha^n = \mathrm{e}^m$. (Again the idea is that we would like to have that $\mathrm{e}^{x+y} = \mathrm{e}^x\,\mathrm{e}^y$.)

This way e^q is defined for all rational numbers.

What about the irrational numbers like $\sqrt{2}$?

Step 2: Next time we will show that

$$E(x+y) = E(x)\,E(y)\,.$$

Here we claim that $E(x) > 0$ for all x. If $x \geq 0$, then this is clear since

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \geq 1\,.$$

If $x < 0$, then using the formula we will show next time we have that

$$1 = E(0) = E(x)\,E(-x)\,.$$

Therefore,

$$E(x) = \frac{1}{E(-x)} > 0\,.$$

Using that $E(x+y) = E(x)\,E(y)$ we now claim that $E(q) = \mathrm{e}^q$ for all rational numbers q.

For integers m this is how we defined e^m. For $m = -k$, where $k \in \mathbf{N}$ we defined

$$\mathrm{e}^{-k} = \frac{1}{\mathrm{e}^k} = \frac{1}{E(k)} = E(-k)\,.$$

For a general rational number $q = \frac{m}{n}$, where $m \in \mathbf{N}$ and $n \in \mathbf{Z}$, we have

$$\left(E\left(\frac{m}{n}\right)\right)^n = E\left(\frac{m}{n}\right)\cdots E\left(\frac{m}{n}\right) = E(m)\,,$$

and

$$E\left(\frac{m}{n}\right) > 0\,.$$

This gives us that $E(q) = \mathrm{e}^q$ for all rational numbers q.

Step 3: We now have $E(x)$ is defined for all x whereas e^x is defined for all rational numbers.

What other properties would we want of the exponential function?

We would want it to be continuous!

Reminder: A function $f : A \to \mathbf{R}$ on some set $A \subset \mathbf{R}$ is said to be continuous if for all $x_0 \in A$ we have:

For all $\epsilon > 0$, there exists a $\delta = \delta(x_0) > 0$ such that if $|x - x_0| < \delta$ ($x \in A$), then $|f(x) - f(x_0)| < \epsilon$.

On Pset 5 you will be asked to show that $E(x)$ is continuous at all points.

Step 4: We will next show that $E(x)$ is the unique continuous function where $E(q) = \mathrm{e}^q$ for all rational numbers q.

Theorem: Let f and g be two continuous function on $\mathbf{R}$ that agrees on all rational numbers, then $f = g$.

We will show this theorem next time. For now here are some more about what it means for a function to be continuous.

Example 1: Suppose $f(x) = c$, where c is a constant. We will show that f is continuous. Given $x_0 \in \mathbf{R}$ and $\epsilon > 0$, set $\delta = 1$. We then have that if $|x - x_0| < \delta = 1$, then $|f(x) - f(x_0)| = 0 < \epsilon$. This show that f is continuous.

Example 2: Suppose $f(x) = x$, we will show that f is continuous. Given $x_0 \in \mathbf{R}$ and $\epsilon > 0$, set $\delta = \epsilon$. We then have that if $|x - x_0| < \delta = \epsilon$, then $|f(x) - f(x_0)| = |x - x_0| < \epsilon$. This show that f is continuous.

Algebraic properties of continuous functions:

- If f and g are continuous functions, then so is $f + g$.
- If f is continuous and c is a constant, then $c\,f$ is continuous.
- If f and g are continuous, then $f\,g$ is also continuous.
- If f is continuous and $f \neq 0$, then $\frac{1}{f}$ is continuous.
- If $f(x)$ and $g(x)$ are continuous, then $f(g(x))$ is continuous.

Proof. (the proof is very similar to the one we gave for the algebraic properties of limits.) □

Theorem: All polynomials are continuous.

Example 3: If $f(x) = x^2 + 1$, then f is continuos. We have already proven that $g(x) = x$ is continuous so by the algebraic properties we have that $x^2 = g\,g$ is continuous. We have also already shown that the constant functions are continuous so $h(x) = 1$ is continuous and therefore by the algebraic properties we have that $f = g^2 + h$ is continuous.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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