SPRING 2025 - 18.100B/18.1002

TOBIAS HOLCK COLDING

Lecture 10

Power series: Suppose that a_n is a sequence. For each x we can form a series

$$\sum_{n=0}^{\infty} a_n\, x^n\,.$$

Exponential function as a power series: Define $E(x)$ by the power series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.$$

Step 0: The power series converges for all x.

Step 1: Define e^q for all rational numbers q.

Step 2: Need to show that

$$E(x+y) = E(x)\, E(y)\,.$$

Step 3: $E(x)$ is defined for all x, whereas e^x is defined for all rational numbers, and $E(q) = \mathrm{e}^q$ for all rational numbers.

Step 4: E is continuous on all of $\mathbf{R}$. (Pset 5.)

Step 5: If f and g are continuous functions on $\mathbf{R}$ that agrees on $\mathbf{Q}$, then $f = g$ everywhere.

Suppose that we have two convergent series

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n$$

of non-negative numbers a_n, $b_n \geq 0$.

Form the "product series"

$$\sum_{n=0}^{\infty} c_n \,,$$

where

$$c_n = \sum_{i=0}^{n} a_i \, b_{n-i} \,.$$

Note that each $c_n \geq 0$ so by the monotone convergence theorem the series

$$\sum_{n=0}^{\infty} c_n$$

is convergent if it is bounded.

Theorem 1: If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are as above, then the series

$$\sum_{n=0}^{\infty} c_n$$

is convergent with limit

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n \,.$$

Proof. Denote

$$s_n^a = \sum_{i=0}^{n} a_i \text{ and } s_n^b = \sum_{i=0}^{n} b_i \text{ and } s_n^c = \sum_{i=0}^{n} c_i \,.$$

The idea here is that

$$(*) \qquad \left(\sum_{\ell=0}^{n} a_\ell\right) \left(\sum_{\ell=0}^{n} b_\ell\right) = \sum_{k=0}^{n} \sum_{i+j=k} a_i \, b_j + \sum_{i+j>n \text{ and } i,j \leq n} a_i \, b_j \leq \sum_{k=0}^{2\,n} \sum_{i+j=k} a_i \, b_j \,.$$

In other words

$$(**) \qquad s_n^c \leq s_n^a \, s_n^b \leq s_{2\,n}^c \,.$$

This is because $(*)$ is

$$s_n^a \, s_n^b = s_n^c + \sum_{i+j>n \text{ and } i,j \leq n} a_i \, b_j \leq s_{2\,n}^c \,,$$

and

$$0 \leq \sum_{i+j>n \text{ and } i,j\leq n} a_i\, b_j\,.$$

Note that the first inequality in $(**)$ implies that the sequence s_n^c is bounded and therefore since a_n, b_n, $c_n \geq 0$ we have that

$$s_n^a \uparrow s^a\,, \qquad s_n^b \uparrow s^b\,, \qquad s_n^c \uparrow s^c$$

by the monotone convergence theorem for sequences. Since the product $s_n^a\, s_n^b$ is squeezed between s_n^c and $s_{2\,n}^c$ by $(**)$ we have that

$$s^c \leq s^a\, s^b \leq s^c\,.$$

From this the claim follows. □

Applying Theorem 1 to the power series $E(x)$ we can now prove the following:

Theorem 2:

$$E(x+y) = E(x)\, E(y)\,.$$

Proof. We will show this assuming that x, $y \geq 0$. Once we have shown the theorem for x, $y \geq 0$ the general case is not too difficult but we will not prove that here. The idea is that $E(x+y)$ will play the role of

$$\sum_{n=0}^{\infty} c_n$$

above. So set

$$c_n = \frac{(x+y)^n}{n!}\,,$$

By the "binomial" formula

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i\, y^{n-i}\,.$$

So

$$c_n = \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} x^i\, y^{n-i}\,.$$

Since

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}\,,$$

we have that

$$c_n = \sum_{i=0}^{n} \frac{x^i}{i!}\, \frac{y^{n-i}}{(n-i)!}\,.$$

This shows that

$$c_n = \sum_{i=0}^{n} a_i\, b_{n-i}\,,$$

where

$$a_i = \frac{x^i}{i!}$$

and

$$b_i = \frac{y^i}{i!}\,.$$

The claim now follows from Theorem 1. □

Coming back to the functions E and e. We have that they agree on all rational numbers and that E is defined for all real numbers.

We would want the exponential function to be continuous!

Reminder: A function $f : A \to \mathbf{R}$ on some set $A \subset \mathbf{R}$ is said to be continuous if for all $x_0 \in A$ we have:

For all $\epsilon > 0$, there exists a $\delta = \delta(x_0) > 0$ such that if $|x - x_0| < \delta$ $(x \in A)$, then $|f(x) - f(x_0)| < \epsilon$.

On Pset 5 you will be asked to show that $E(x)$ is continuous at all points.

Step 5: We will show that $E(x)$ is the unique continuous function where $E(q) = \mathrm{e}^q$ for all rational numbers q.

Theorem 3: (On Pset 5.) Let f and g be two continuous function on $\mathbf{R}$ that agrees on all rational numbers, then $f = g$.

We will next see that there are functions on $\mathbf{R}$ that are not continuous at any point!

Before defining such a function recall that we already proved that $\sqrt{2}$ is a irrational number and thus for all $\delta > 0$, there exists an N such that if $n \geq N$, then

$$0 < \frac{\sqrt{2}}{n} < \delta\,.$$

So arbitrarily close to zero there are irrational numbers. Likewise by the Archimedean property we have that arbitrarily close to any irrational number there is a rational number.

On $\mathbf{R}$ define a function f as follows

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & \text{otherwise} \end{cases}$$

We claim that f is nowhere continuous. Suppose first that x_0 is rational and let $0 < \epsilon < 1$. We have that $f(x_0) = 1$ and for any $\delta > 0$, there exists a irrational number x with $|x - x_0| < \delta$ but we also have that

$$\epsilon < 1 = |f(x) - f(x_0)| \, .$$

This show that f is discontinuous at x_0.

Likewise suppose x_0 is an irrational number. We have that $f(x_0) = 0$. Given $0 < \epsilon < 1$ for any $\delta > 0$, there exists a rational number x with $|x - x_0| < \delta$. On the other hand

$$\epsilon < 1 = |f(x) - f(x_0)| \, .$$

This show that f is discontinuous at x_0.

This gives an example of a function that is discontinuous at all points. On the other hand recall from last time how to generate continuous functions from known continuous functions:

Algebraic properties of continuous functions:

- If f and g are continuous functions, then so is $f + g$.
- If f is continuous and c is a constant, then $c\, f$ is continuous.
- If f and g are continuous, then $f\, g$ is also continuous.
- If f is continuous and $f \neq 0$, then $\frac{1}{f}$ is continuous.
- If $f(x)$ and $g(x)$ are continuous, then $f(g(x))$ is continuous.

Proof. (The proof is very similar to the one we gave for the algebraic properties of limits of sequences.) □

Theorem: All polynomials are continuous.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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MIT, Dept. of Math., 77 Massachusetts Avenue, Cambridge, MA 02139-4307.

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