SPRING 2025 - 18.100B/18.1002

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Lecture 11

Reminder: A function $f : A \to \mathbf{R}$ on some set $A \subset \mathbf{R}$ is said to be continuous if for all $x_0 \in A$ we have:

For all $\epsilon > 0$, there exists a $\delta = \delta(x_0) > 0$ such that if $|x - x_0| < \delta$ ($x \in A$), then $|f(x) - f(x_0)| < \epsilon$.

Two theorems about continuous functions:

Extreme Value Theorem: Suppose that $f : [a, b] \to \mathbf{R}$ is a continuous function, then there exist $x_M \in [a, b]$ such that $f(x_M) \geq f(x)$ for all $x \in [a, b]$. Similarly, there exists $x_m \in [a, b]$ such that $f(x_m) \leq f(x)$ for all $x \in [a, b]$.

Intermediate Value Theorem: Suppose that $f : [a, b] \to \mathbf{R}$ is a continuous function, then for all y between $f(a)$ and $f(b)$, there exists $x \in [a, b]$ such that $f(x) = y$.

We will show these theorems using a lemma that connects sequences and continuous functions. This is the following:

Lemma: Suppose that $f : [a, b] \to \mathbf{R}$ is a continuous function and $x_n \to x_\infty$ a sequence, then $f(x_n) \to f(x_\infty)$. We can also write this as

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) .$$

Proof. To show that $f(x_n) \to f(x_\infty)$ let $\epsilon > 0$ be given. Since f is continuous at x_∞, there exists $\delta > 0$ such that if $|x - x_\infty| < \delta$, then $|f(x) - f(x_\infty)| < \epsilon$. Since $x_n \to x_\infty$ there exists N such that if $n \geq N$, then $|x_n - x_\infty| < \delta$ and therefore $|f(x_n) - f(x_\infty)| < \epsilon$. This shows the lemma. $\square$

Using this lemma we can now prove the extreme value theorem:

Proof. (of EVT.) Let $E = f([a,b])$ and set $M = \sup E$. We will show that $M < \infty$ and that $M = f(x)$ for some $x \in [a,b]$. We show first that M is finite. Otherwise for each n there exists an $x_n \in [a,b]$ such that $f(x_n) > n$. Since the sequence $\{x_n\}$ is bounded by the Bolzano-Weirstrass theorem it has a convergent subsequence. Let us denote that by x_{n_k}. We have $x_{n_k} \to x_\infty \in [a,b]$. By the lemma above $f(x_{n_k}) \to f(x_\infty)$ but we assumed that the sequence $f(x_{n_k})$ is unbounded which is the desired contradiction.

For each integer n we can now choose $x_n \in [a,b]$ such that $f(x_n) > M - \frac{1}{n}$. Again since this sequence is bounded by the Bolzano-Weirstrass theorem it has a convergent subsequence $x_{n_k} \to x_\infty \in [a,b]$. By the lemma above $f(x_{n_k}) \to f(x_\infty) \geq M$. Since $M = \sup f([a,b])$ we have that $f(x_\infty) = M$. This show the EVT. $\square$

Proof. (of IVT.) We will assume that $f(a) < 0 < f(b)$ and show that there exists $x \in [a,b]$ such that $f(x) = 0$. The general case is similar. Let $A = \{y \mid \text{ for all } x \leq y \text{ we have that } f(x) \leq 0\}$. Note that $a \in A$ so the set is non-empty. Set $M = \sup A$ and let x_n be a sequence with $x_n < M$ and $x_n \to M$. It follows that $f(x_n) < 0$ and so by the lemma above we must have that $f(M) \leq 0$. We are done if $f(M) = 0$ so assume that $f(M) < 0$. We have that $M < b$ and by continuity there exist a whole interval around M where $f < 0$. This contradict that M was the supremum of the set A. Showing the IVT. $\square$

Abstract metric space.

Definition: Metric space A metric space is a set X with a function $d : X \times X \to \mathbf{R}$ with the following three properties:

(1) $d(x, y) \geq 0$ for all x, $y \in X$ and $d(x, y) = 0$ if and only if $x = y$. (Distances ≥ 0.)
(2) $d(x, y) = d(y, x)$. (Symmetric.)
(3) $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality.)

Examples:

(1) $X = \mathbf{R}$ and
$$d(x, y) = |x - y| \,.$$
(2) $X = \mathbf{R}^2$ and for $x = (x_1, x_2)$ and $y = (y_1, y_2)$
$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \,.$$
(3) $X = \mathbf{R}^3$ and for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$
$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2} \,.$$

Example: Continuous function on an interval $[a, b]$. Let $X = C([a, b])$ where $C([a, b])$ is the set of continuous functions on $[a, b]$. The distance between two continuous functions f and g is then
$$d(f, g) = \max_{x \in [a,b]} |f(x - g(x)| \,.$$
Since $f - g$ is also a continuous function the EVT theorem guarantees that the max is achieved for some $x \in [a, b]$.

Metric spaces plays the role of generalised real numbers. A lot of the discussion that we have had in the class holds also for metric spaces and this is useful in many circumstances. For instance, we will see in a later class that we can use it to solve ODEs.

Sequences in a metric space: A sequence in a metric space (X, d) is a map $f : \mathbf{N} \to X$. We typically denote the image $f(n)$ by x_n. Similarly we define a **subsequence** as the composition of a strictly increasing map $g : \mathbf{N} \to \mathbf{N}$ with f and $x_{n_k} = f(g(k))$.

It is not all results that we know from $\mathbf{R}$ that generalises to general metric spaces. For instance, in general there are no algebraic properties, no squeeze theorem, no monotone convergence theorem. On the other hand the statement of both the Cauchy convergence theorem and the Bolzano-Weirstrass theorems makes sense in a general metric space.

Example (Box distance): The space is $X = \mathbf{R}^2$ and if $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$, then

$$d(\underline{x}, \underline{y}) = |x_1 - y_1| + |x_2 - y_2|\,.$$

Example (Strange metric on integers): The space is $X = \mathbf{N}$ and if m, n are integers, then

$$d(m, n) = \frac{1}{n} - \frac{1}{m} \quad .$$

Here is a wild example of a metric space:

Example (French railway metric): The space is $X = \mathbf{R}^2$ and if $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$, then

$$d(\underline{x}, \underline{y}) = \begin{cases} |\underline{x} - \underline{y}| & \text{if } \underline{x} = c\,\underline{y} \text{ or } \underline{y} = c\,\underline{x} \text{ for some } c \in \mathbf{R} \\ |\underline{x}| + |\underline{y}| & \text{otherwise} \end{cases} .$$

Here

$$|\underline{x} - \underline{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\,,$$

and likewise for $|\underline{x}|$ and $|\underline{y}|$.

Definition: **Convergent sequence in a metric space** If (X, d) is a metric space and x_n is a sequence in X, then we say that x_n converges to x and write $x_n \to x$ or $x = \lim_{n\to\infty} x_n$ if for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then

$$d(x, x_n) < \epsilon\,.$$

This is equivalent to that the sequence $d(x_n, x_\infty) \to 0$.

Definition: **Cauchy sequence in a metric space** If (X, d) is a metric space and x_n is a sequence in X, then we say that x_n is a Cauchy sequence if for all $\epsilon > 0$, there exists an N, such that if m, $n \geq N$, then

$$d(x_m, x_n) < \epsilon\,.$$

Theorem: In any metric space (X, d) a convergent sequence is also a Cauchy sequence.

Proof. So suppose that $x_n \in X$ is a sequence and $x_n \to x$. Given $\epsilon > 0$, convergence means that there exists N such that if $n \geq N$, then $d(x, x_n) < \frac{\epsilon}{2}$. If both m, $n \geq N$, then we have by the triangle inequality that

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

This show the theorem. □

The converse is not always the case: If $X = (0, 1) \subset \mathbf{R}$ with $d(x, y) = |x - y|$, then the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence but since 0 is not in X, it is not convergent. We sometimes express this by saying that in this case X is not Cauchy complete.

Definition: Continuous function on a metric space (X, d) Suppose that $F : X \to \mathbf{R}$ is a function. We say that f is continuous at $x_0 \in X$, if for all $\epsilon > 0$, there exists a $\delta > 0$, such that if $x \in X$ with $d(x, x_0) < \delta$, then

$$|F(x) - F(x_0)| < \epsilon\,.$$

Example: Let again $X = C([0, 1])$ be the set of continuous functions on $[0, 1]$. Equip X with the distance described above. So the distance between to continuous functions f and g is then

$$d(f, g) = \max_{x \in [a,b]} |f(x - g(x)|\,.$$

Define F on X to be the function $F(f) = f(0)$ where $f \in C([0, 1])$. F is easily seen to be a continuous function on the metric space X.

We can now extend one of the earlier lemmas to general metric spaces.

Lemma: Let (X, d) be a general metric space. Suppose that $f : X \to \mathbf{R}$ is a continuous function and x_n is a sequence in X with $x_n \to x_\infty$, then $f(x_n) \to f(x_\infty)$.

Proof. To show that $f(x_n) \to f(x_\infty)$ let $\epsilon > 0$ be given. Since f is continuous at x_∞, there exists $\delta > 0$ such that if $d(x, x_\infty) < \delta$, then $|f(x) - f(x_\infty)| < \epsilon$. Since $x_n \to x_\infty$ there exists N such that if $n \geq N$, then $d(x_n, x_\infty) < \delta$ and therefore $|f(x_n) - f(x_\infty)| < \epsilon$. This shows the lemma. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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