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Lecture 12

Review (discussed in lectures so far):

(1) $\mathbf{R}$ is the complete ordered field that contains $\mathbf{Q}$.
(2) Sequences and limits.
(3) Series.
(4) Continuous functions.
(5) Metric spaces.

Definition: Metric space. A metric space is a set X with a function $d : X \times X \to \mathbf{R}$ with the following three properties:

(1) $d(x, y) \geq 0$ for all x, $y \in X$ and $d(x, y) = 0$ if and only if $x = y$. (Distances ≥ 0.)
(2) $d(x, y) = d(y, x)$. (Symmetric.)
(3) $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality.)

Examples (Euclidean distance):

(1) $X = \mathbf{R}$ and
$$d(x, y) = |x - y| \, .$$
(2) $X = \mathbf{R}^2$ and for $x = (x_1, x_2)$ and $y = (y_1, y_2)$
$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \, .$$
(3) $X = \mathbf{R}^3$ and for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$
$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2} \, .$$

Example: Continuous function on an interval $[a, b]$. Let $X = C([a, b])$ where $C([a, b])$ is the set of continuous functions on $[a, b]$. The distance between two continuous functions f and g is then
$$d(f, g) = \max_{x \in [a,b]} |f(x - g(x)| \, .$$
Since $f - g$ is also a continuous function the EVT theorem guarantees that the max is achieved for some $x \in [a, b]$.

Example (Box distance): The space is $X = \mathbf{R}^2$ and if $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$, then

$$d(\underline{x}, \underline{y}) = |x_1 - y_1| + |x_2 - y_2| \,.$$

Sequences in a metric space: A sequence in a metric space (X, d) is a map $f : \mathbf{N} \to X$. We typically denote the image $f(n)$ by x_n. Similarly we define a **subsequence** as the composition of a strictly increasing map $g : \mathbf{N} \to \mathbf{N}$ with f and $x_{n_k} = f(g(k))$.

It is not all results that we know from $\mathbf{R}$ that generalises to general metric spaces. For instance, in general there are no algebraic properties, no squeeze theorem, no monotone convergence theorem. On the other hand the statement of both the Cauchy convergence theorem and the Bolzano-Weirstrass theorems makes sense in a general metric space.

Definition: Convergent sequence in a metric space If (X, d) is a metric space and x_n is a sequence in X, then we say that x_n converges to x and write $x_n \to x$ or $x = \lim_{n\to\infty} x_n$ if for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then

$$d(x, x_n) < \epsilon \,.$$

This is equivalent to that the sequence $d(x_n, x) \to 0$.

Definition: Cauchy sequence in a metric space If (X, d) is a metric space and x_n is a sequence in X, then we say that x_n is a Cauchy sequence if for all $\epsilon > 0$, there exists an N, such that if m, $n \geq N$, then

$$d(x_m, x_n) < \epsilon \,.$$

Theorem: In any metric space (X, d) a convergent sequence is also a Cauchy sequence.

Proof. So suppose that $x_n \in X$ is a sequence and $x_n \to x$. Given $\epsilon > 0$, convergence means that there exists N such that if $n \geq N$, then $d(x, x_n) < \frac{\epsilon}{2}$. If both m, $n \geq N$, then we have by the triangle inequality that

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \,.$$

This show the theorem. □

The converse is not always the case: If $X = (0, 1) \subset \mathbf{R}$ with $d(x, y) = |x - y|$, then the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence but since 0 is not in X, it is not convergent. We sometimes express this by saying that in this case X is not Cauchy complete.

A metric space is said to be Cauchy complete if every Cauchy sequence is convergent.

Definition: (metric) ball. If (X, d) is a metric space, $x \in X$ and $r > 0$, then

$$B_r(x) = \{y \in X \mid d(x, y) < r\}$$

is said to be the ball with center x and radius r.

Definition: Bounded subset. If (X, d) is a metric space and $A \subset X$, then we say that A is bounded if A is contained in some metric ball $B_r(x)$.

Theorem: In a metric space (X, d) any Cauchy sequence is bounded.

Proof. Suppose that x_n is a Cauchy sequence. By definition of a Cauchy sequence, there exists some N such that if m, $n \geq N$, then

$$d(x_n, x_m) < 1\,.$$

Set

$$r = 1 + \max\{d(x_N, x_i) \mid i < N\}\,.$$

We claim that

$$\{x_n\} \subset B_r(x_N)\,.$$

Since $r \geq 1$ and $d(x_N, x_n) < 1$ for $n \geq N$ we only need to see that $x_n \in B_r(x_N)$ for $n < N$. This follows from that $d(x_N, x_n) < r$ when $n < N$ by definition of r. □

Bolzano - Weirstrass theorem: Any bounded sequence of real numbers have a convergent subsequence. This theorem does not hold for a general metric space but it holds if the metric space is compact. To discuss this we need the notion of what an open subset of a metric space is.

Definition (Open subset): Let (X, d) be a metric space. We say that O is an open subset of X if for all $x \in O$, there exists an $r > 0$ such that $B_r(x) \subset O$.

Note that $\emptyset$ (the empty set) and X are both open.

On subsets of a set X we have the following operations.

- Union of two or more subsets.
  If U_1 and U_2 are subsets, then $U_1 \cup U_2$ is the union. So
  $$U_1 \cup U_2 = \{x \in X \,|\, x \in U_1 \text{ or } x \in U_2 \text{ or both}\}\,.$$
  Similarly, for union of more than two subsets.
- Intersection of two or more subsets.
  If U_1 and U_2 are subsets, then $U_1 \cap U_2$ is the intersection. So
  $$U_1 \cap U_2 = \{x \in X \,|\, x \in U_1 \text{ and } x \in U_2\}\,.$$
  Similarly, for intersection of more than two subsets.
- Complement of a subset U.
  $X \setminus U$ is all the elements of X that are not in U.

Example: $X = \mathbf{R}$, $A = (0,3)$, $B = (-1,2)$ and $C = (0,2)$.
$A \cup B = (0,3)$.
$A \cap B = (0,2)$.
$X \setminus A = (-\infty, 0] \cup [3, \infty)$.
$C \subset B$.

Union and intersections of families of subsets

- Union of families.
  If U_α is a family of subsets, then $\cup_\alpha U_\alpha$ is the union of all the subsets. So
  $$\cup_\alpha U_\alpha = \{x \in X \,|\, x \in U_\alpha \text{ for some } \alpha\}\,.$$
- Intersection of families.
  If U_α is a family of subsets, then $\cap_\alpha U_\alpha$ is the intersection of all the subsets. So
  $$\cap_\alpha U_\alpha = \{x \in X \,|\, x \in U_\alpha \text{ for all } \alpha\}\,.$$

Example: $X = \mathbf{R}$, $U_n = (-\frac{1}{n}, \frac{1}{n})$, where $n \in \mathbf{N}$, then
$$\cup_n U_n = (-1,1) \text{ and } \cap_n U_n = \{0\}\,.$$

Lemma: For a set X and subsets A, B we have $A = B$ if and only if $A \subset B$ and $B \subset A$.

Lemma: For a set and subset A, B and A_α we have

(1) $X \setminus (X \setminus A) = A$.
(2) $X \setminus \cup_\alpha A_\alpha = \cap_\alpha (X \setminus A_\alpha)$.
(3) $X \setminus \cap_\alpha A_\alpha = \cup_\alpha (X \setminus A_\alpha)$.

Proof. To prove the first of these claim that $X \setminus (X \setminus A) = A$ we need to show two directions. Suppose $x \in A$, then $x \notin X \setminus A$ and therefore $x \in X \setminus (X \setminus A)$. Conversely, if $x \in X \setminus (X \setminus A)$, then $x \notin X \setminus A$ and therefore $x \in A$.

To prove the second claim observe that if $x \in X \setminus \cup_\alpha A_\alpha$, then $x \notin \cup_\alpha A_\alpha$ so x is not in any of the A_α's. Therefore x must be in all the $X \setminus A_\alpha$ and hence in the intersection of those so $x \in \cap_\alpha (X \setminus A_\alpha)$. This show that $X \setminus \cap_\alpha A_\alpha \subset \cap_\alpha (X \setminus A_\alpha)$. To show the other direction suppose that $x \in \cap_\alpha (X \setminus A_\alpha)$. This means that for all α we have that $x \notin A_\alpha$. Therefore, $x \notin \cup_\alpha A_\alpha$ and hence $x \in X \setminus (\cup_\alpha A_\alpha)$. This show the other direction.

Finally, to prove the third claim observe that if $x \in X \setminus \cap_\alpha A_\alpha$, then $x \notin \cap_\alpha A_\alpha$ and so there exists some α so that $x \in X \setminus A_\alpha$. In other words, $x \in \cup_\alpha (X \setminus A_\alpha)$. This show one direction. To see the other direction observe that if $x \in \cup_\alpha (X \setminus A_\alpha)$, then there exists some α so that $x \in X \setminus A_\alpha$. It follows that $x \notin A_\alpha$ and hence $x \notin \cap_\alpha A_\alpha$ but instead $x \in X \setminus \cap_\alpha A_\alpha$. This show the other direction and completes the proof of the lemma. ☐

Lemma: In a metric space any ball $B_r(x)$ is an open subset.

Proof. Suppose that $y \in B_r(x)$, and let $s = r - d(x, y)$. Note that since $y \in B_r(x)$ we have that $d(x, y) < r$ and so $s > 0$. We will show that $B_s(y) \subset B_r(x)$. To see that assume that $z \in B_s(y)$ we then have that $d(y, z) < s$ and so by the triangle inequality

$$d(z, x) \leq d(z, y) + d(y, x) < s + d(y, x) = (r - d(x, y)) + d(y, x) = r\,.$$

This shows the claim. ☐

Lemma: In a metric space if O_α are open subsets, then

$$\cup_\alpha O_\alpha$$

is open.

Proof. See Pset. ☐

Lemma: In a metric space if $O_1, \cdots, O_n$ are finitely many open subsets, then

$$O_1 \cap \cdots \cap O_n$$

is open.

Proof. Suppose that $x \in O_1 \cap \cdots \cap O_n$, then x lies in each O_i. For each i, there exists an $r_i > 0$, such that $B_{r_i}(x) \subset O_i$. Let $r = \min r_i$, then for each i we have that $B_r(x) \subset B_{r_i}(x) \subset O_i$ so $B_r(x)$ is a subset of each O_i and hence $B_r(x) \subset O_1 \cap \cdots \cap O_n$. This shows the claim. ☐

Warning: Intersection of infinitely many open subsets may not be open!!!!

Example: $X = \mathbf{R}$ and for each natural number let O_n be the open set $O_n = (-\frac{1}{n}, \frac{1}{n})$, then

$$\cap_n O_n = \{0\}\,.$$

So the intersection of these infinitely many open subsets is not open.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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