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Lecture 13

Definition (Closed subsets): Let (X, d) be a metric space. We say that C is a closed subset of X if the complement $X \setminus C$ is open.

Note that $\emptyset$ (the empty set) and X are both closed.

Examples:
- $(0, 1)$ is not a closed subset of $\mathbf{R}$.
- $\{0\}$ is a closed subset of $\mathbf{R}$.
- $[0, 1]$ is a closed subset of $\mathbf{R}$.
- $[0, 1] \times [0, 1]$ is a closed subset of $\mathbf{R}^2$.

Lemma: Let (X, d) be a metric space and $r > 0$, then

$$A_r = \{y \,|\, d(x, y) > r\}$$

is open. Equivalently, $\bar{B}_r(x) = \{y \,|\, d(x, y) \leq r\}$ is closed.

Proof. Suppose that $y \in A_r$, then $d(y, x) > r$ and if we set $s = d(y, x) - r$, then $s > 0$. Moreover, if $z \in B_s(y)$, then by the triangle inequality

$$d(x, y) \leq d(y, z) + d(z, x)\,.$$

So

$$r < r + s - d(y, z) \leq d(x, y) - d(y, z) \leq d(z, x)\,.$$

This show that $B_s(z) \subset A_r$ and so A_r is open. □

There is an equivalent way of defining closed subsets and that comes from the next theorem.

Theorem: A subset C of a metric space (X, d) is closed if and only if for all convergent sequences x_n with all x_n in C also the limit is in C.

Proof. Suppose first that A is closed and let x_n be a convergent sequence is A with limit x we need to show that $x \in A$. Since A is closed the complement is open and if $x \in X \setminus A$, then there exists some $r > 0$ so $B_r(x) \subset X \setminus A$ and therefore for all $y \in A$ we would have that $d(y, x) \geq r$. This contradict that $x_n \to x$ and $x_n \in A$.

We also need to show the converse. So suppose that A is a subset with the property that for all sequences in A that are convergent in X the limit is in A. We will show that A is closed or equivalent that the complement is open. If the complement is not open, then there exists an $x \in X \setminus A$ such that no ball around x is entirely contained in the complement. Therefore for each n there exists an $x_n \in A$. This sequence converges to x which was assumed not to be in A contradicting that A contained all limits of sequences in A and therefore the complement must be open and A itself closed. □

For union and intersection of closed subsets we have the following:

Theorem:

- Union: If C_α is a family of closed subsets, then $\cap_\alpha C_\alpha$ is also closed.
- Intersection: If $C_1, \cdots, C_n$ are closed subsets, then $C_1 \cup \cdots \cup C_n$ is also closed.

Proof. There are several ways of proving this. The easiest is probably straight from the definition using the operations on sets. For the first claim we need to show that the complement of $\cap_\alpha C_\alpha$ is open. Using the operations of sets we have that

$$X \setminus \cap_\alpha C_\alpha = \cup_\alpha (X \setminus C_\alpha)\,.$$

Since each $X \setminus C_\alpha$ are open this is the union of open sets and therefore open. This shows the first claim.

To see the second claim we argue similarly. We want to show that $C_1 \cup \cdots \cup C_n$ is closed or, equivalently, $X \setminus (C_1 \cup \cdots \cup C_n)$ is open. However,

$$X \setminus (C_1 \cup \cdots \cup C_n) = (X \setminus C_1) \cap \cdots \cap (X \setminus C_n)\,,$$

where the last is the intersection of finitely many open sets and therefore open. This show the second claim. □

Warning: Union of infinitely many closed sets may not be closed!!!

Definition (Cover, open cover and finite sub-cover): If A is a subset of X, then a cover of A is a collection collection of subsets U_α of X so that

$$A \subset \cup_\alpha U_\alpha\,.$$

We say that a $U_{\alpha_1}, \cdots, U_{\alpha_n}$ is a finite sub-cover if also $\{U_{\alpha_i}\}_i$ is a cover.

If (X, d) is a metric space and all the U_α are open, then we say that $\{U_\alpha\}_\alpha$ is an open cover.

Example: If $X = \mathbf{R} \times \mathbf{R}$, then $A_n = (-n, n) \times (-n, n)$ is an open cover of X.

Example: Note that in the example where $X = \mathbf{R} \times \mathbf{R}$ and $A_n = (-n, n) \times (-n, n)$ is an open cover, then there is no finite sub-cover. On the other hand if $A \subset \mathbf{R} \times \mathbf{R}$ is bounded, then for n sufficiently large $A \subset A_n$ so for A, there is a finite sub-cover of this cover.

Definition (Compact subset): If (X, d) is a metric space and A is a subset, then we say that A is compact if each open cover has a finite sub-cover.

Example: If (X, d) is $\mathbf{R}$ with the usual metric and $A = (0, 1)$, then $A_n = (\frac{1}{n}, 1)$ is an open cover of A but there is no finite sub-cover of $\{A_n\}_n$ that covers A.

Theorem: $[a, b] \subset \mathbf{R}$ is compact.

Proof. We will show this next time. □

Theorem: If (X, d) is a metric space and A a compact subset, then A is closed and bounded.

Proof. Suppose first that A is not closed. We will show that this leads to a contradiction. If it is not closed, then there exists a convergent sequence $x_n \in A$ with limit x not in A. Set

$$O_n = \left\{ y \,|\, d(x, y) > \frac{1}{n} \right\}$$

. By the earlier lemma these are open sets. Since $\cup_n A_n = X \setminus \{x\}$ and x is assumed not to be in A we indeed have that A_n is an open cover of A. Since $A_n \subset A_{n+1}$ any finite cover of A_n's would be contained in A_N for some large N but this would imply that for all $y \in A$ we would have that $d(x, y) > \frac{1}{N}$ contradicting that $x_n \in A$ and $x_n \to x$. This show that the limit x is in A.

Since A is compact,

$$X = \cup_y B_r(y)$$

and each $B_1(x)$ is open, then finitely many of these covers A. Say $A \subset B_1(y_1) \cup \cdots \cup B_1(y_n)$. Set $r = 1 + \max_i\{d(y_1, y_i)\}$. It follows by the triangle inequality that $A \subset B_r(y_1)$. Hence, A is bounded. □

Warning: The converse is not the case!!! There are closed a bounded subsets of metric spaces that are not compact.

If $(X, d) = (0, 1)$ with the usual metric, then X is closed and bounded but it is not compact.

Here is a more illuminating example:

Example: Let $X = C([0, 1])$ be the set of continuous functions on the unit interval $[0, 1]$. We equip X with the metric where

$$d(f, g) = \max_x |f(x) - g(x)| .$$

Let $f_n(x)$ be the sequence of continuous functions on $[0, 1]$ given by that

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{n+1} \\ 1 - n\,(n+1)\,\left(x - \frac{1}{n+1}\right) & \text{if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

We have the f_n is a bounded sequence. After all they all lies in the metric ball $B_2(0)$ where 0 is the zero function. That is, the function on $[0, 1]$ that is identically equal to zero. However, the sequence f_n does not have a convergent subsequence (and does not even have a subsequence that is a Cauchy sequence). Indeed, for any $m \neq m$ we have that

$$d(f_m, f_n) = 1 .$$

Note also that the (closed) ball $A = \bar{B}_1(0)$ is closed and bounded but not compact. It is not compact because for the balls $\cup_f B_{\frac{1}{2}}(f)$ finitely many does not cover A. If finitely many did cover A, then for one such ball say $B_{\frac{1}{2}}(f)$ infinitely many f_n's would lie in it but any two elements in such a ball would have distance < 1 showing that there could at most be one f_n in such a ball.

Theorem: If (X, d) is a metric space and A a compact subset, then any closed subset C contained in A is also compact.

Proof. Let O_α be a open cover of C. Since C is closed $X \setminus C$ is open and so $\{O_\alpha\}$ together with $X \setminus C$ is an open cover of A and hence finitely many of those say $O_1, \cdots, O_n, X \setminus C$ covers A. Since $X \setminus C$ contains no elements in C it follows that $C \subset O_1 \cup \cdots \cup O_n$ and thus C is compact. □

Bolzano-Weirstrass theorem for metric spaces.

Theorem: If (X, d) is a metric space and A a compact subset, then any sequence in A has a convergent subsequence.

Proof. We will show this next time. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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