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Lecture 14

Definition (Compact subset): If (X,d) is a metric space and A is a subset, then we say that A is compact if each open cover has a finite sub-cover.

Theorem 0: If (X,d) is a metric space and A a compact subset, then A is closed and bounded.

Warning: The converse is not the case!!! There are closed a bounded subsets of metric spaces that are not compact.

Example: If $(X,d)=(0,1)$ with the usual metric, then X is closed and bounded but it is not compact.

Here is a more illuminating example:

Example: Let $X=C([0,1])$ be the set of continuous functions on the unit interval $[0,1]$. We equip X with the metric where

$$d(f,g)=\max_x |f(x)-g(x)|\,.$$

Let $f_n(x)$ be the sequence of continuous functions on $[0,1]$ given by that

$$f_n(x)=\begin{cases} 1 & \text{if } 0\leq x\leq \frac{1}{n+1} \\ 1-n\,(n+1)\,\left(x-\frac{1}{n+1}\right) & \text{if } \frac{1}{n+1}\leq x\leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

We have the f_n is a bounded sequence. After all they all lies in the metric ball $B_2(0)$ where 0 is the zero function. That is, the function on $[0,1]$ that is identically equal to zero. However, the sequence f_n does not have a convergent subsequence (and does not even have a subsequence that is a Cauchy sequence). Indeed, for any $m \neq m$ we have that

$$d(f_m, f_n) = 1\,.$$

Note also that the (closed) ball $A = \bar{B}_1(0)$ is closed and bounded but not compact. It is not compact because for the balls $\cup_f B_{\frac{1}{2}}(f)$ finitely many does not cover A. If finitely many did cover A, then for one such ball say $B_{\frac{1}{2}}(f)$ infinitely many f_n's would lie in it but any two elements in such a ball would have distance < 1 showing that there could at most be one f_n in such a ball.

Using what we have shown in earlier lectures one can show the following:

Theorem 1: In $\mathbf{R}^n$, a subset is compact if and only if it is closed and bounded.

In a general metric space this is not the case as the above examples shows.

We won't show this theorem here but instead we will show a version of the Bolzano-Weirstrass theorem for metric spaces. This is the next theorem.

Theorem 2: If (X, d) is a metric space and A a compact subset, then any sequence in A has a convergent subsequence.

Before proving Theorem 2 we will need some results:

Lemma: Let (X, d) be a compact metric space if C_α is a family of closed (decreasing) nested subsets. That is, closed subsets so that $C_{n+1} \subset C_n$. If all C_n are non-empty, then

$$\cap_n C_n \neq \emptyset\,.$$

Proof. Set $O_\alpha = X \setminus C_\alpha$, then each O_α is open. If $\cap_n C_n \neq \emptyset$, then

$$\cup_n O_\alpha = X\,.$$

Therefore, finitely many of the O_n's cover X by compactness. Denote these by O_i for $i = 1, \cdots, k$. Since

$$O_1 \cup \cdots \cup O_k = X$$

it follows that

$$C_1 \cap \cdots \cap C_k = \emptyset\,.$$

However, by the nested property one of these k closed subsets is the smallest, say C_k and therefore $C_1 \cap \cdots \cap C_k = C_k$. Contradicting that the intersection is empty. $\square$

Before stating the next results recall that in a metric space (X, d) the set $\bar{B}_r(x) = \{y \in X \mid d(x, y) \leq r\}$ is closed and is referred to as the closed ball. The above lemma gives the following useful corollary:

Corollary: Let (X, d) be a compact metric space and suppose that $B_{r_n}(x_n)$ is a family of balls with centres x_n and radii $r_n > 0$, where $r_n \to 0$ and $\bar{B}_{r_{n+1}}(x_{n+1}) \subset \bar{B}_{r_n}(x_n)$. Then

$$\cap_n \bar{B}_{r_n}(x_n) = \{x\}\,.$$

That is, the intersection is non-empty and consists of a single point.

Proof. Set

$$A = \cap_n \bar{B}_{r_n}(x_n)\,.$$

Observe first that for each n we have that $x_n \in \bar{B}_{r_n}(x_n)$ so from the lemma above we have that A is non-empty. We claim that A consists of just one element. Suppose that $x,\, y \in A$, for any integer n we have that

$$x,\, y \in \bar{B}_{r_n}(x_n)\,,$$

and so by the triangle inequality

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \leq r_n + r_n = 2\, r_n\,.$$

Since this holds for all n we see that $d(x, y) = 0$ and so there is at most one such point. $\square$

Proof. (of Theorem 2.) Suppose that x_n is a sequence in a compact subset A of a metric space. Fix $r > 0$ and write

$$A \subset_{x \in A} B_r(x)\,.$$

Since A is compact finitely many of these cover A. This means that in one of these balls, say $B_r(y_1)$, there are infinitely many x_n's. From here on and out we will focus on this ball. Since $A \cap \bar{B}_r(y)$ is a closed subset of a compact set we can now cover $\bar{B}_r(y)$ by balls of radius $\frac{r}{4}$. By compactness finitely many of these sub-balls cover the ball $\bar{B}_r(y)$. In one of those sub-balls there are also infinitely many x_n's. Fix such a sub-ball and call it $\bar{B}_{\frac{r}{4}}(y_2)$. We have that

$$\bar{B}_{\frac{r}{2}}(y_2) \subset \bar{B}_{2\,r}(y_1)$$

and that infinitely many x_n's belongs to $B_{\frac{r}{4}}(y_2)$. If the original $r = 1$ gives after repeating this process i times balls $B_{4^{1-i}}(y_i)$ so that

$$\cdots \subset \bar{B}_{2\,4^{1-i}}(y_i) \subset \cdots \bar{B}_{2\,4^{-1}}(y_2) \subset \bar{B}_2(y_1)\,.$$

where each of these balls contains infinitely many elements from the original sequence. Since the radii of this sequence converges to zero this sequence satisfies the assumptions of the corollary we have from the corollary that

$$\cap B_{2\,4^{1-i}}(y_i) = \{x\}\,.$$

Moreover, we can pick a subsequence x_{n_k} of the original sequence such that

$$x_{n_k} \subset \bar{B}_{2\,4^{1-k}}(y_k)\,.$$

It follows that this subsequence converges to x. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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