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Lecture 15

Definition: If $f : \mathbf{R} \to \mathbf{R}$ is a function, then we say that f is differentiable at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. (Note that in this fraction x is assumed to be $\neq x_0$.) When the limit exists, then we say that the function f is differentiable at x_0 and that its derivative at x_0 is the limit. In this case we denote the derivative at x_0 by $f'(x_0)$.

Examples:

(1) Constant functions. Suppose that $f(x) = c$ for some constant $c \in \mathbf{R}$, then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{c - c}{x - x_0} = 0\,.$$

It follows that the limit exists and is zero and so f is differentiable at all points and the derivative is zero.

(2) Linear functions. Suppose that $f(x) = x$, then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x - x_0}{x - x_0} = 1\,.$$

It follows that the limit exists and is one and so f is differentiable at all points and the derivative is one.

These are just two examples where we computed the derivative directly from the definition. How do we compute the derivative of a general function?

For that there are some tools:

- Sum rule.
- Product rule.
- Quotient rule.
- Chain rule.

Once we know how to compute the derivative of a function, then we would like to understand the function using information about its derivative. For that we have the following tools:

- Mean value theorem.
- L'Hopital's rule.
- Taylor expansion.

Before getting to how to use the derivative we need to be able to compute it. For that it will be useful to note the following:

Lemma: If f is differentiable at x_0, then f is continuous at x_0.

Proof. Since f is differentiable at x_0 we have that

$$\frac{f(x)-f(x_0)}{x-x_0} \to f'(x_0)\,.$$

Therefore, there exist $\delta_1 > 0$ such that if $|x-x_0| < \delta_1$, then

$$\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) \;< 1$$

or, equivalently,

$$|f(x)-f(x_0)-f'(x_0)\,(x-x_0)| < |x-x_0|\,.$$

Therefore, for $|x-x_0| < \delta_1$ we have

$$|f(x)-f(x_0)| < (|f'(x_0)|+1)\,|x-x_0|\,.$$

Given $\epsilon > 0$, set

$$\delta = \min\left\{\delta_1, \frac{\epsilon}{|f'(x_0)|+1}\right\}\,.$$

It follows that if $|x-x_0| < \delta$, then

$$|f(x)-f(x_0)| < \epsilon\,.$$

This show that f is continuous at x_0. □

Example: On the real line suppose that f is the function given by that $f(0) = 0$ and for all other x

$$f(x) = x\,\sin\frac{1}{x}\,.$$

This is an example of a function that is continuous at zero but not differentiable at zero. It is not differentiable at zero because it fluctuate too much near zero. To see that it is continuous at zero we will use that $|\sin t| \leq 1$ for all t. Indeed using that it is easy to see that f is continuous at zero. Next we will see that it is not differentiable at zero. To see that we look at the difference quotient

$$\frac{f(x)-f(0)}{x-0}=\frac{x\,\sin\frac{1}{x}-0}{x-0}=\sin\frac{1}{x}\,.$$

As $x\to 0$ this function fluctuate between -1 and 1 so it does not have a limit and therefore the original function f is not differentiable at zero.

Example: If we dampen the fluctuation of the function given in the previous example further, then we get a differentiable function at zero even if it still fluctuate but just not as much. This is done in the following example. Suppose that f is the function that is given by that $f(0)=0$ and for all other x

$$f(x)=x^2\,\sin\frac{1}{x}\,.$$

Again we form the difference quotient

$$\frac{f(x)-f(0)}{x-0}=\frac{x^2\,\sin\frac{1}{x}-0}{x-0}=x\,\sin\frac{1}{x}\,.$$

In this case we see that as $x\to 0$, then $x\,\sin\frac{1}{x}\to 0$ and so the function is differentiable at zero and the derivative there is zero.

The following is very useful to compute the derivative of many functions:

Theorem: If f, g are functions on $\mathbf{R}$ that both are differentiable at x_0, then

- (Sum rule.)
$$(f+g)'(x_0)=f'(x_0)+g'(x_0)\,.$$
- (Leibniz's rule.)
$$(f\,g)(x_0)=f'(x_0)\,g(x_0)+f(x_0)\,g'(x_0)\,.$$
- (Quotient rule.) If also $g(x_0)\neq 0$, then
$$\left(\frac{f}{g}\right)'(x_0)=\frac{f'(x_0)\,g(x_0)-f(x_0)\,g'(x_0)}{g^2(x_0)}\,.$$

Proof. To prove the sum rule consider the difference quotient

$$\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}\to f'(x_0)+g'(x_0)$$

This show the sum rule.

To prove the Leibniz rule we form the difference quotient

$$\frac{(f\,g)(x)-(f\,g)(x_0)}{x-x_0}\,.$$

We rewrite this using a trick we have used before in other settings. Namely, we can write this as

$$\frac{(f\,g)(x)-(f\,g)(x_0)}{x-x_0}=\frac{f(x)\,g(x)-f(x)\,g(x_0)+f(x)\,g(x_0)-f(x_0)\,g(x_0)}{x-x_0}$$
$$=f(x)\,\frac{g(x)-g(x_0)}{x-x_0}+\frac{f(x)-f(x_0)}{x-x_0}\,g(x_0)\to f(x_0)\,g'(x_0)+f'(x_0)\,g(x_0)\,.$$

(Here we used that by the continuity lemma above $f(x)\to f(x_0)$.) This proves Leibniz's rule.

Finally, to prove the quotient rule we observe first that since g is differentiable at x_0 it is continuous at x_0 and therefore (since $g(x_0)\neq 0$) when x is close to x_0 we have that $g(x)\neq 0$. Moreover, we have that

$$\frac{\frac{f(x)}{g(x)}-\frac{f(x_0)}{g(x_0)}}{x-x_0}=\frac{f(x)\,g(x_0)-f(x_0)\,g(x)}{(x-x_0)\,g(x)\,g(x_0)}$$
$$=\frac{f(x)\,g(x_0)-f(x_0)\,g(x_0)}{(x-x_0)\,g(x)\,g(x_0)}+\frac{f(x_0)\,g(x_0)-f(x_0)\,g(x)}{(x-x_0)\,g(x)\,g(x_0)}$$
$$=\frac{1}{g(x)}\,\frac{f(x)-f(x_0)}{x-x_0}+\frac{f(x_0)}{g(x)\,g(x_0)}\,\frac{g(x)-g(x)}{x-x_0}$$
$$\to\frac{f'(x_0)}{g(x_0)}+\frac{f(x_0)\,g'(x_0)}{g^2(x_0)}\,.$$

From this the claim easily follows. □

Leibniz's rule is named after Gottfried Wilhelm Leibniz (1646 - 1716). Leibniz [from Wikipedia] was a German polymath active as a mathematician, philosopher, scientist and diplomat who is credited, alongside Sir Isaac Newton, with the creation of calculus in addition to many other branches of mathematics, such as binary arithmetic and statistics. Leibniz has been called the "last universal genius" due to his vast expertise across fields, which became a rarity after his lifetime with the coming of the Industrial Revolution and the spread of specialised labor.

Theorem: (Chain rule.) If $f:[a,b]\to[c,d]$ and $g:[c,d]\to\mathbf{R}$ are functions, where f is differentiable at x_0 and g differentiable at $y_0=f(x_0)$, then the composition $g\circ f$ is differentiable at x_0 and the derivative at x_0 is

$$(g\circ f)'(x_0)=g'(y_0)\,f'(x_0)\,.$$

Proof. Set $y=f(x)$ and $y_0=f(x_0)$. Assume first that $f'(x_0)\neq 0$. In this case for $x\neq x_0$ but close to x_0 we have that $y\neq y_0$ and we can write the difference quotient as follows. We have that

$$\frac{g(f(x))-g(f(x_0))}{x-x_0}=\frac{g(y)-g(y_0)}{y-y_0}\,\frac{f(x)-f(x_0)}{x-x_0}\,.$$

Since f is differentiable at x_0 as $x \to x_0$ we have that

$$\frac{f(x) - f(x_0)}{x - x_0} \to f'(x_0)\,.$$

Moreover, when $x \to x_0$ we have that $f(x) = y \to f(x_0) = y_0$ by the continuity lemma above. It follows that when $x \to x_0$ we have that

$$\frac{g(y) - g(y_0)}{y - y_0} \to g'(y_0)\,.$$

Combining this gives that

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(y) - g(y_0)}{y - y_0}\,\frac{f(x) - f(x_0)}{x - x_0} \to g'(y_0)\,f'(x_0)\,.$$

This proves the chain rule when $f'(x_0) \neq 0$. When $f'(x_0) = 0$ we argue as above but have to be more careful as in this case we can have that $y = y_0$ even when $x \neq x_0$. For x where $y = y_0$ the difference quotient is zero and where $y \neq y_0$ we can argue as above and rewrite the difference quotient as the product of two factors. In either case we get that the limit is zero proving the remaining case of the chain rule. □

Lemma: Let $f : [a, b] \to \mathbf{R}$ be a differentiable function and suppose that $a < x_0 < b$ and that f has a local maximum or minimum at x_0, then

$$f'(x_0) = 0\,.$$

Proof. Suppose that x_0 is a local maximum. The proof when x_0 is a local minimum. It follows from the assumption that for all x near x_0

$$f(x) - f(x_0) \leq 0\,.$$

Therefore, when $x > x_0$ we have that

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0\,,$$

whereas when $x < x_0$ we have that for the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0\,.$$

Since the limit is the same whether x converges to x_0 from the the left (negative side) or from the right (positive side) it follows that $f'(x_0) = 0$ as claimed.

□

Theorem: (Rolle's theorem.) Let $f : [a, b] \to \mathbf{R}$ be a differentiable function with $f(a) = f(b)$, then there exists a x_0 between a and b such that

$$f'(x_0) = 0\,.$$

Proof. There are three cases to consider:

(1) f is constant equal to $f(a)$.
(2) For some x between a and b we have that $f(x) > f(a)$.
(3) For some x between a and b we have that $f(x) < f(a)$.

In the first case the function is constant and the derivate is zero everywhere. The second and third cases are similar so we will just argue in the second case. In the second case by the extreme value theorem there exists some x_0 such that $f(x_0) = \max f > f(a)$. It now follows from the previous lemma that $f'(x_0) = 0$. □

Theorem: (Mean value theorem.) Let $f : [a, b] \to \mathbf{R}$ be a differentiable function, then there exists a x_0 between a and b such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}\,.$$

.

Proof. Consider the function g given by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}\,(x - a)\,.$$

Observe that for g we have $g(a) = g(b)$ and so Rolle's theorem applies and we have that there exists some x_0 where $g'(x_0) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}\,,$$

the claim follows. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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