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Lecture 16

Last time we defined what it means for a function to be differentiable. This is the following:

Definition: If $f : \mathbf{R} \to \mathbf{R}$ is a function, then we say that f is differentiable at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. (Note that in this fraction x is assumed to be $\neq x_0$.) When the limit exists, then we say that the function f is differentiable at x_0 and that its derivative at x_0 is the limit. In this case we denote the derivative at x_0 by $f'(x_0)$.

One of the first things we showed about differentiable function was that they are continuous:

Lemma: If f is differentiable at x_0, then f is continuous at x_0.

We also established some very useful rules for computing the derivative of functions that are constructed from other functions whose derivative we know:

Theorem: If f, g are functions on $\mathbf{R}$ that both are differentiable at x_0, then

- (Sum rule.)
$$(f+g)'(x_0) = f'(x_0) + g'(x_0)\,.$$
- (Leibniz's rule.)
$$(f\,g)(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)\,.$$
- (Quotient rule.) If also $g(x_0) \neq 0$, then
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{g^2(x_0)}\,.$$

Finally, for the composition of functions we have the chain rule:

Theorem: (Chain rule.) If $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbf{R}$ are functions, where f is differentiable at x_0 and g differentiable at $y_0 = f(x_0)$, then the composition $g \circ f$ is differentiable at x_0 and the derivative at x_0 is

$$(g \circ f)'(x_0) = g'(y_0)\, f'(x_0)\,.$$

Now that we know how to compute the derivative of many functions we will be interested in using the derivative to describe the growth or decay of a function. The first step towards this is the next lemma.

Before stating it recall that a function $f : \mathbf{R} \to \mathbf{R}$ has a local maximum at x_0 if there exists a $\delta > 0$ such that

$$f(x_0) = \max_{[x_0-\delta, x_0+\delta]} f\,,$$

and similarly for a local minimum.

Lemma: Let $f : [a, b] \to \mathbf{R}$ be a differentiable function and suppose that $a < x_0 < b$ and that f has a local maximum or minimum at x_0, then

$$f'(x_0) = 0\,.$$

Proof. Suppose that x_0 is a local maximum. The proof when x_0 is a local minimum is similar. It follows from the assumption that for all x near x_0

$$f(x) - f(x_0) \leq 0\,.$$

Therefore, when $x > x_0$ we have that

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0\,,$$

whereas when $x < x_0$ we have that for the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0\,.$$

Since the limit is the same whether x converges to x_0 from the the left (negative side) or from the right (positive side) it follows that $f'(x_0) = 0$ as claimed. $\square$

We can now use this lemma to establish the following very useful result:

Theorem: (Rolle's theorem.) Let $f : [a,b] \to \mathbf{R}$ be a differentiable function with $f(a) = f(b)$, then there exists a x_0 between a and b such that

$$f'(x_0) = 0\,.$$

Proof. There are three cases to consider:

(1) f is constant equal to $f(a)$.
(2) For some x between a and b we have that $f(x) > f(a)$.
(3) For some x between a and b we have that $f(x) < f(a)$.

In the first case the function is constant and the derivate is zero everywhere. The second and third cases are similar so we will just argue in the second case. In the second case by the extreme value theorem there exists some x_0 such that $f(x_0) = \max f > f(a)$. It now follows from the previous lemma that $f'(x_0) = 0$. □

Rolle's theorem can then be used to show both the mean value theorem and the Cauchy mean value theorem:

Theorem: (Mean value theorem.) Let $f : [a,b] \to \mathbf{R}$ be a differentiable function, then there exists a x_0 between a and b such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}\,.$$

.

Proof. Consider the function g given by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}\,(x - a)\,.$$

Observe that for g we have $g(a) = g(b)$ and so Rolle's theorem applies and we have that there exists some x_0 where $g'(x_0) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}\,,$$

the claim follows. □

Theorem: (Cauchy mean value theorem.) Let f, $g : [a,b] \to \mathbf{R}$ be differentiable functions, then there exists a x_0 between a and b such that

$$f'(x_0)\,[g(b) - g(a)] = g'(x_0)\,[f(b) - f(a)]\,.$$

In particular, if $g(b) - g(a) \neq 0$, then

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}\,.$$

Proof. Consider the function
$$h(x) = f(x)\,[g(b) - g(a)] - g(x)\,[f(b) - f(a)]\,.$$
Note that
$$h(a) = f(a)\,[g(b) - g(a)] - g(a)\,[f(b) - f(a)] = f(a)\,g(b) - g(a)\,f(b)\,.$$
$$h(b) = f(b)\,[g(b) - g(a)] - g(b)\,[f(b) - f(a)] = f(a)\,g(b) - g(a)\,f(b)\,.$$
Therefore, by Rolle's theorem, there exists x_0 between a and b such that $h'(x_0) = 0$. Since
$$h'(x) = f'(x)\,[g(b) - g(a)] - g'(x)\,[f(b) - f(a)]$$
this shows the claim. □

We observe that the Cauchy mean value theorem implies the earlier mean value theorem. Namely, if we let the second function g be $g(x) = x$, then $g'(x) = 1$ and $g(b) - g(a) = b - a$. Therefore, the Cauchy mean value theorem becomes
$$f'(x_0)\,(b - a) = f'(x_0)\,(g(b) - g(a)) = g'(x_0)\,(f(b) - f(a)) = f(b) - f(a)\,,$$
which is the earlier mean value theorem.

The next two rules are useful to establishing the limit of a faction of function when the denominator either tend to zero or infinity.

Theorem: (L'Hopital's rule, version 1.) Let $f,\ g : (a, b) \to \mathbf{R}$ be differentiable functions with $g(x) \neq 0$ and $g'(x) \neq 0$ for all x, assume that
$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0\,.$$
If
$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$
exists, then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}\,.$$

Proof. We will see that this is an easy consequence of the Cauchy mean value theorem. By assumption given $\epsilon > 0$, there exists $\delta > 0$ such that if $a < x < \delta$, then
$$\frac{f'(x)}{g'(x)} - L \;< \epsilon\,.$$
By the Cauchy mean value theorem we have for any y with $a < y < x$ that there exist z with $y < z < x$ so that
$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)}\,.$$

We therefore have that

$$\frac{f(x)-f(y)}{g(x)-g(y)} - L \; < \epsilon\,.$$

By letting $y \to 0$ we see that

$$\frac{f(x)}{g(x)} - L \; \leq \epsilon\,.$$

Since this holds for all ϵ we get that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L$$

as claimed. $\square$

Theorem: (L'Hopital's rule, version 2.) Let $f,\ g : (a,b) \to \mathbf{R}$ be differentiable functions with $g(x) \neq 0$ and $g'(x) \neq 0$ for all x, assume that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty\,.$$

If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}\,.$$

Proof. Given $\epsilon > 0$, since $\frac{f'(x)}{g'(x)} \to L$ as as $x \to a$ we have that there exists a $\delta > 0$ such that if $a < x < a + 2\,\delta$, then

$$\frac{f'(x)}{g'(x)} - L \; < \epsilon\,.$$

Set $x_1 = a + \delta$. For a given $x \in (a, x_1)$, there exists $x_0 \in (x, x_1)$ such that

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(x_1)-f(x)}{g(x_1)-g(x)}\,.$$

It follows that

$$\frac{f(x_1)-f(x)}{g(x_1)-g(x)} - L \; < \epsilon\,.$$

By dividing the nominator and denominator of the fraction in this expression by $g(x)$ we get

$$\frac{\frac{f(x)}{g(x)} - \frac{f(x_1)}{g(x)}}{1 - \frac{g(x_1)}{g(x)}} - L \; < \epsilon\,.$$

This implies that

$$\frac{f(x)}{g(x)} - \frac{f(x_1)}{g(x_1)} - L\left(1 - \frac{g(x_1)}{g(x)}\right) \; < \epsilon\left(1 - \frac{g(x_1)}{g(x)}\right)\,.$$

Since this holds for all $x \in (a, a+\delta)$ and $g(x) \to \infty$ as $x \to a$ we have that for $x > a$ but sufficiently close to a that

$$\left|\frac{f(x)}{g(x)} - L\right| \leq \epsilon\,.$$

Since this holds for all ϵ we see that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

This proves the claim. □

Finally, we have the following key fact that show that any differentiable function can be approximated by a polynomial and give a way of estimate the difference between the function and the approximating polynomial.

Theorem: (Taylor expansion.) Let $f : [a, b] \to \mathbf{R}$ be a function and k a positive integer. Assume that f, f', $f^{(2)}, \cdots, f^{(k-1)}$ exists on $[a, b]$ and are continuous and that $f^{(k)}$ is defined on (a, b), then there exists c between a and b such that

$$f(b) = f(a) + f'(a)\,(b-a) + \frac{f^{(2)}(a)}{2}\,(b-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}\,(b-a)^{k-1}$$

$$+\frac{f^{(k)}(c)}{(k)!}\,(b-a)^k\,.$$

Proof. Define the Taylor polynomial by

$$P(x) = f(a) + f'(a)\,(x-a) + \frac{f^{(2)}(a)}{2}\,(x-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}\,(x-a)^{k-1}$$

and define a number M by that

$$f(b) = P(b) + \frac{M}{k!}\,(b-a)^k\,.$$

We want to show that there exists some c between a and b such that

$$M = f^{(k)}(c)\,.$$

To do that we set

$$R(x) = f(x) - P(x) - \frac{M}{k!}\,(x-a)^k\,.$$

We have that $R(a) = R(b) = 0$ and so by Rolle's theorem, there exists some c_1 between a and b with $R'(c_1) = 0$. Next observe that $R'(a) = R'(c_1) = 0$ and so again by Rolle's theorem, there exists c_2 between a and c_1 with $R^{(2)}(c_2) = 0$. Since $R^{(i)}(a) = 0$ for $i = 0, \cdots, k-1$ we can continue this process k times and find some $c = c_k$ such that $R^{(k)}(c) = 0$. However, $0 = R^{(k)}(c) = f^k(c) - M$ Therefore, $M = f^{(k)}(c)$ as claimed. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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