SPRING 2025 - 18.100B/18.1002

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Lecture 17

Recall that last time we showed the Taylor expansion theorem:

Theorem: (Taylor expansion.) Let $f : [a, b] \to \mathbf{R}$ be a function and k a positive integer. Assume that f, f', $f^{(2)}, \cdots, f^{(k-1)}$ exists on $[a, b]$ and are continuous and that $f^{(k)}$ is defined on (a, b), then there exists c between a and b such that

$$f(b) = f(a) + f'(a)\,(b-a) + \frac{f^{(2)}(a)}{2}\,(b-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}\,(b-a)^{k-1}$$

$$+\frac{f^{(k)}(c)}{(k)!}\,(b-a)^k\,.$$

For and infinitely differentiable function f on $\mathbf{R}$ we define the $(k-1)$ Taylor polynomial at a by

$$P_{k-1}(x) = f(a) + f'(a)\,(x-a) + \frac{f^{(2)}(a)}{2}\,(x-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}\,(x-a)^{k-1}\,.$$

Question: One naturally wonders how well does this polynomial approximate f when x is near a?

Answer: This depend on the value of the remainder

$$R_k(x) = \frac{f^{(k)}(c)}{k!}\,(x-a)^k\,.$$

Example 1: Suppose that $f(x) = \mathrm{e}^x$ so $f^{(k)}(x) = f(x)$ for all k. This means that the Taylor expansion near $a = 0$ becomes

$$P_{k-1}(x) = \sum_{i=0}^{k-1} \frac{x^i}{i!}\,.$$

By the Taylor expansion theorem we have that

$$f(x) = P_{k-1}(x) + \frac{f^{(k)}(c)}{k!}\, x^k\,.$$

Since $f^{(k)}(x) = f(x)$ for all k, it follows from the Taylor expansion theorem that we have

$$|f(x) - P_{k-1}(x)| \leq \frac{\mathrm{e}^{|x|}}{k!}\,.$$

We conclude that for k large the polynomial P_{k-1} gives a pretty good approximation to f. For instance, if $|x| \leq 1$, then we have that

$$|f(x) - P_{k-1}(x)| \leq \frac{\mathrm{e}}{k!}\,.$$

Example 2: On $\mathbf{R}$ define a function f by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \mathrm{e}^{-\frac{1}{x^2}} & \text{otherwise} \end{cases}$$

It is easy to see that f is infinitely differentiable and that $f^{(k)}(0) = 0$ for all k. It follows that for all k the Taylor polynomial at 0 is $P_{k-1} \equiv 0$. Thus in this case $f(x) = R_k(x)$.

Riemann integrals

Partition: Let $[a, b]$ be an interval. A partition $\mathcal{P}$ of the interval $[a, b]$ is a number of sub-divisions x_i such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b\,.$$

The partition is then the sub-intervals $[x_{i-1}, x_i]$. We will set $\Delta\, x_i = x_i - x_{i-1}$.

Upper and lower sums: Suppose now that $f : [a, b] \to \mathbf{R}$ is a bounded function and that $\mathcal{P} = \{x_i\}$ is a partition of the interval $[a, b]$. We define upper and lower sums as follows. Set

$$M_i = \sup_{[x_{i-1},x_i]} f\,,$$

$$m_i = \inf_{[x_{i-1},x_i]} f\,,$$

and upper $U(f,\mathcal{P})$ and lower sums $L(f,\mathcal{P})$ by

$$U(f,\mathcal{P}) = \sum_{i=1}^{n} M_i \, \Delta\, x_i \,,$$

$$L(f,\mathcal{P}) = \sum_{i=1}^{n} m_i \, \Delta\, x_i \,.$$

Example 3: Suppose that the function is $f(x) = x^2 + 1$ on the interval $[-2, 2]$ and that the partition is $\mathcal{P}$ is $\{-2, -1, 0, 1, 2\}$. We have

$$m_1 = 2 \text{ and } M_1 = 5\,,$$
$$m_2 = 1 \text{ and } M_2 = 2\,,$$
$$m_3 = 1 \text{ and } M_3 = 2\,,$$
$$m_4 = 2 \text{ and } M_4 = 5\,.$$

For the lower and upper sums we have

$$L(f,\mathcal{P}) = 2 + 1 + 1 + 2 = 6\,,$$
$$U(f,\mathcal{P}) = 5 + 2 + 2 + 5 = 14\,.$$

The following lemma is immediate (from that $M_i \geq m_i$):

Lemma 1: We always have that

$$U(f,\mathcal{P}) \geq L(f,\mathcal{P})\,.$$

Sub-partition: Let $[a, b]$ be an interval and $\mathcal{P}_1$ and $\mathcal{P}_2$ two partitions of the interval $[a, b]$. We say that $\mathcal{P}_2$ is a sub-partition (or refinement) of $\mathcal{P}_1$ if all the dividing points in $\mathcal{P}_1$ are also in $\mathcal{P}_1$ (and then presumable some additional dividing points).

Example 4: Suppose that the interval is $[-2, 2]$ and the given partition $\mathcal{P}_1$ is

$$\{-2, -1, 0, 1, 2\}\,.$$

Then the partition

$$\mathcal{P}_2 = \left\{-2, -1\,\frac{1}{2}, -1, 0, \frac{1}{2}, 1, 2\right\}$$

is a refinement (or sub-division) of $\mathcal{P}_1$. Indeed, $\mathcal{P}_2$ has the same dividing points as $\mathcal{P}_1$ in addition to some more.

We now have the following:

Lemma 2: Suppose now that $f : [a, b] \to \mathbf{R}$ is a bounded function and that $\mathcal{P}_1$ is a partition of the interval $[a, b]$ and $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$, then

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_1)\,.$$

Proof. The middle inequality is the previous lemma. The inequality to the right follows from that if $\mathcal{P}_2$ is a subdivision of $\mathcal{P}_1$. Namely, suppose that $a = x_0 < x_1 < \cdots < x_n = b$ are the dividing points for $\mathcal{P}_1$ and that between say x_{i-1} and x_i there is an extra dividing point in $\mathcal{P}_2$ say y so $x_{i-1} < y < x_i$, then we have

$$\sup_{[x_{i-1},y]} f \leq M_i$$

and

$$\sup_{[y,x_i]} f \leq M_i$$

so

$$[\sup_{[x_{i-1},y]} f]\,(y - x_{i-1}) + [\sup_{[y,x_i]} f]\,(x_i - y) \leq M_i\,\Delta\,x_i\,.$$

From this it follows easily that

$$U(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_1)\,.$$

Similarly, for the inequality to the left. □

Upper and lower integrals: Suppose now that $f : [a, b] \to \mathbf{R}$ is a bounded function. Define the upper Riemann integral of f by

$$\overline{\int_a^b} f\,dx = \inf_{\mathcal{P}} U(f, \mathcal{P})\,.$$

Here the infimum is taken over all partitions of $[a, b]$. Likewise, we define the lower Riemann integral by

$$\underline{\int_a^b} f\,dx = \sup_{\mathcal{P}} L(f, \mathcal{P})\,.$$

Riemann integral: Suppose that $f : [a, b] \to \mathbf{R}$ is a bounded function, then we say that f is Riemann integrable if

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx\,.$$

If the function is Riemann integrable, then the Riemann integral is

$$\int_a^b f\,dx = \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx\,.$$

The Riemann integrable functions is denoted by $\mathcal{R}\,([a,b])$.

From Wikipedia: Georg Friedrich Bernhard Riemann (1826 – 1866) was a German mathematician who made profound contributions to analysis, number theory, and differential geometry. Riemann held his first lectures in 1854, which founded the field of Riemannian geometry and thereby set the stage for Albert Einstein's general theory of relativity. In the field of real analysis, he is mostly known for the first rigorous formulation of the integral, the Riemann integral, and his work on Fourier series. His contributions to complex analysis include most notably the introduction of Riemann surfaces, breaking new ground in a natural, geometric treatment of complex analysis. His 1859 paper on the prime-counting function, containing the original statement of the Riemann hypothesis, is regarded as a foundational paper of analytic number theory. He is considered by many to be one of the greatest mathematicians of all time.

Example 5: Let $f : [0,1] \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \cap \mathbf{Q} \\ 1 & \text{otherwise} \end{cases}$$

For this function and all partitions $\mathcal{P}$ we have that

$$L(f,\mathcal{P}) = 0 \text{ and } U(f,\mathcal{P}) = 1\,.$$

Thus, f is not Riemann integrable.

We will be interested in the questions: "What kind of functions are Riemann integrable?"

.... and "How do we compute the integral?"

The answer to the second question will be the fundamental theorem of calculus. This will be the topic of a later lecture.

Lemma 3: Suppose now that $f : [a,b] \to \mathbf{R}$ is a bounded function, then $f \in \mathcal{R}([a,b])$ if and only if for all $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon\,.$$

Proof. Suppose that $f \in \mathcal{R}([a,b])$, then

$$\sup_{\mathcal{P}} L(f,\mathcal{P}) = \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx = \inf_{\mathcal{P}} U(f,\mathcal{P})\,.$$

This means that given $\epsilon > 0$, there exists partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$\int_a^b f\,dx - \frac{\epsilon}{2} < L(f,\mathcal{P}_1)$$

and

$$U(f,\mathcal{P}_2) \leq \int_a^b f\,dx + \frac{\epsilon}{2}\,.$$

Let $\mathcal{P}$ be the partition that has all the dividing points of both $\mathcal{P}_1$ and $\mathcal{P}_2$. So $\mathcal{P}$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$. It follows that

$$\int_a^b f\,dx - \frac{\epsilon}{2} < L(f,\mathcal{P}_1) \leq L(f,\mathcal{P}) \leq U(f,\mathcal{P}) \leq U(f,\mathcal{P}_2) \leq \int_a^b f\,dx + \frac{\epsilon}{2}\,.$$

This proves the claim.

To see the converse, suppose that for some $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon\,.$$

Since

$$L(f,\mathcal{P}) \leq \underline{\int_a^b} f\,dx$$

and

$$\overline{\int_a^b} f\,dx \leq U(f,\mathcal{P})$$

we have that

$$\overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon\,.$$

Since this holds for all $\epsilon > 0$ we get the claim. □

We now get to a key theorem that gives a simple criterium for a function to be Riemann integrable:

Theorem: Any continuous function on $[a,b]$ is in $\mathcal{R}([a,b])$.

Proof. We will show this next time once we have shown that a continuous function on a closed and bounded interval is, in fact, uniformly continuous. □

The proof of this theorem needs the following key concept.

Definition: Uniformly continuous. Suppose that $f : I \to \mathbf{R}$ is a function, where I is an interval. We say that f is uniformly continuous if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ if } |x - y| < \delta\,.$$

Note that being uniformly continuous is stronger than being continuous. It means that for a given $\epsilon > 0$, the same δ can be used for all x.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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