SPRING 2025 - 18.100B/18.1002

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Lecture 18

Definition: Uniformly continuous. Suppose that $f : I \to \mathbf{R}$ is a function, where I is an interval. We say that f is uniformly continuous if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ if } |x - y| < \delta\,.$$

Note that being uniformly continuous is stronger than being continuous. It means that for a given $\epsilon > 0$, the same δ can be used for all x.

Example 1: Suppose that

$$f(x) = x^2$$

on $\mathbf{R}$, then f is NOT uniformly continuous. To see this, let $\epsilon > 0$ be given if f was uniformly continuous, then there would exists $\delta > 0$ such that

$$f(x + \delta) - f(x) < \epsilon\,,$$

for all x. This would mean that

$$2\,\delta\,x < (x + \delta)^2 - x^2 < \epsilon$$

for all x, which is clearly not the case.

Example 2: Suppose that

$$f(x) = \frac{1}{x}$$

on $(0, 1]$, then f is NOT uniformly continuous. To see this, consider $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2\,n}$, then

$$|f(x_n) - f(y_n)| = n$$

and

$$|x_n - y_n| < \frac{1}{n}\,.$$

From this it easily follows that f is not uniformly continuous.

Theorem 1: Any continuous function on $[a, b]$ is uniformly continuous.

Proof. Suppose not; then there exists $\epsilon > 0$ such that for all $n > 0$, there are x_n and y_n with

$$|x_n - y_n| < \frac{1}{n}$$

and so that

$$|f(x_n) - f(y_n)| \geq \epsilon\,.$$

Since the interval $[a, b]$ is compact we can choose a subsequence of x_n say x_{n_k} so that

$$x_{n_k} \to x\,.$$

Since

$$|x - y_{n_k}| \leq |x - x_{n_k}| + |x_{n_k} - y_{n_k}|$$

we have that $y_{n_k} \to x$ as well. Since f is continuous we have that $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$. However, this contradict that

$$|f(x_{n_k} - f(y_{n_k})| \geq \epsilon\,,$$

. □

We now get to a key theorem that gives a simple criterium for a function to be Riemann integrable:

Theorem 2: Any continuous function on $[a, b]$ is in $\mathcal{R}\,([a, b])$.

Proof. Given $\epsilon > 0$, since f is uniformly continuous by Theorem 1 it follows that there exists $\delta > 0$ such that if $|x - y| < \delta$, then

$$|f(x) - f(y)| < \frac{\epsilon}{b - a}\,.$$

Let $\mathcal{P}$ be a partition so that for all i we have $\Delta\, x_i < \delta$, then on each interval of the partition of the form $[x_{i-1}, x_i]$ we have that

$$M_i - m_i < \frac{\epsilon}{b - a}\,.$$

It follows that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_i M_i\,\Delta\, x_i - \sum_i m_i\,\Delta\, x_i$$

$$= \sum_i [M_i - m_i]\,\Delta\, x_i < \frac{\epsilon}{b - a} \sum_i \Delta\, x_i = \epsilon\,.$$

Since this holds for all $\epsilon > 0$ we have that f is integrable. □

Basic properties of integrals.

Theorem 3: We have the following basic formulas for integrals:

(1) If $f \in \mathcal{R}([a,b])$ and $c \in \mathbf{R}$, then $c\, f \in \mathcal{R}([a,b])$ and

$$\int_a^b (c\, f)\, dx = c \int_a^b f\, dx\,.$$

(2) If $f,\, g \in \mathcal{R}([a,b])$, then $f+g \in \mathcal{R}([a,b])$ and

$$\int_a^b (f+g)\, dx = \int_a^b f\, dx + \int_a^b g\, dx\,.$$

(3) If $f,\, g \in \mathcal{R}([a,b])$ and $f \leq g$, then

$$\int_a^b f\, dx \leq \int_a^b g\, dx\,.$$

(4) If $f \in \mathcal{R}([a,b])$ and $c \in (a,b)$, then $f \in \mathcal{R}([a,c])$ and $f \in \mathcal{R}([c,b])$ and

$$\int_a^c f\, dx + \int_c^b f\, dx = \int_a^b f\, dx\,.$$

Proof. The first claim follow from that if $\mathcal{P}$ is a partition, then

$$L(c\, f, \mathcal{P}) = c\, L(f, \mathcal{P})$$

and

$$U(c\, f, \mathcal{P}) = c\, U(f, \mathcal{P})\,.$$

To prove the second claim. Given $\epsilon > 0$, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions so that

$$U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1) < \frac{\epsilon}{2}$$

and

$$U(g, \mathcal{P}_2) - L(g, \mathcal{P}_2) < \frac{\epsilon}{2}\,.$$

Let $\mathcal{P}$ be the partition that has the combined dividing points of $\mathcal{P}_1$ and $\mathcal{P}_2$. It follows that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \frac{\epsilon}{2}$$

and

$$U(g, \mathcal{P}) - L(g, \mathcal{P}) < \frac{\epsilon}{2}\,.$$

Therefore,

$$U(f+g, \mathcal{P}) - L(f+g, \mathcal{P}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

From this the second claim follows.

To see the third claim let $\mathcal{P}$ be any partition of $[a,b]$. It follows that

$$U(g, \mathcal{P}) \leq U(f, \mathcal{P})\,.$$

Since

$$\int_a^b g\, dx = \inf_{\mathcal{P}} U(g, \mathcal{P})$$

and likewise for f the claim now follows.

Finally, to see the fourth claim. Let $\mathcal{P}$ be any partition of $[a,b]$ and let $\mathcal{P}_0$ be the refinement of $\mathcal{P}$ that in addition to the dividing points of $\mathcal{P}$ also have c as a dividing point. It follows that

$$U(f,\mathcal{P}_0\cap[a,c])+U(f,\mathcal{P}_0\cap[c,b])=U(f,\mathcal{P}_0)\,.$$

Likewise,

$$L(f,\mathcal{P}_0\cap[a,c])+L(f,\mathcal{P}_0\cap[c,b])=L(f,\mathcal{P}_0)\,.$$

Therefore,

$$\begin{aligned}U(f,\mathcal{P}_0\cap[a,c])-L(f,\mathcal{P}_0\cap[a,c])+U(f,\mathcal{P}_0\cap[c,b])-L(f,\mathcal{P}_0\cap[c,b])\\=U(f,\mathcal{P}_0)-L(f,\mathcal{P}_0)\,.\end{aligned}$$

From this the fourth claim easily follows. □

Corollary: Suppose that f, $|f|\in\mathcal{R}([a,b])$, then

$$\int_x^b f\,dx \;\leq \int_a^b |f|\,dx\,.$$

Proof. This follows from the lemma since $f\leq|f|$ and $-f\leq|f|$. Namely, from the first of these inequalities together with the lemma we get that

$$\int_a^b f\,dx\leq\int_a^b |f|\,dx\,,$$

whereas from the second we get that

$$-\int_a^b f\,dx=\int_a^b(-f)\,dx\leq\int_a^b|f|\,dx\,.$$

Together these gives the claim. □

Fundamental theorem of calculus, version 1: Let f be a continuous function on $[a,b]$ and define F on $[a,b]$ by

$$F(x)=\int_a^x f(s)\,ds\,.$$

The function F is differentiable with derivative f.

Proof. Fix $x_0\in[a,b]$ and assume first that $x>x_0$. We then have that

$$F(x)=\int_a^x f(s)\,ds=\int_a^{x_0} f(s)\,ds+\int_{x_0}^x f(s)\,ds=F(x_0)+\int_{x_0}^x f(s)\,ds\,.$$

It follows that

$$F(x)-F(x_0)=\int_{x_0}^x f(s)\,ds\,.$$

Therefore,

$$(x-x_0)\,\min_{[x_0,x]}f\leq F(x)-F(x_0)\leq(x-x_0)\,\max_{[x_0,x]}f$$

and hence

$$\min_{[x_0,x]} f \leq \frac{F(x)-F(x_0)}{x-x_0} \leq \max_{[x_0,x]} f\,.$$

Since f is continuous at x_0 as $x \to x_0$ both the left and right hand side of this string of inequalities converges to $f(x_0)$. This proves the claim when $x > x_0$. When $x < x_0$ we can write $F(x)$ as

$$F(x) + \int_x^{x_0} f(s)\,ds = F(x_0)\,.$$

Therefore,

$$F(x) - F(x_0) = -\int_x^{x_0} f(s)\,ds\,.$$

Arguing as above gives the claim also in this case. □

Fundamental theorem of calculus, version 2: Suppose that $F:[a,b]\to \mathbf{R}$ is differentiable and that $F' = f \in \mathcal{R}\left([a,b]\right)$, then

$$F(b) - F(a) = \int_a^b f(s)\,ds\,.$$

Proof. Since f is integrable, then for all $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon\,.$$

For a given partition $\mathcal{P}$ with dividing points x_i we have

$$L(f,\mathcal{P}) = \sum_i m_i\,(x_i - x_{i-1})\,,$$

$$U(f,\mathcal{P}) = \sum_i M_i\,(x_i - x_{i-1})\,,$$

Moreover, by the mean value inequality

$$F(x_i) - F(x_{i-1}) = f(y_i)\,(x_i - x_{i-1})\,.$$

We now have that

$$m_i\,(x_i - x_{i-1}) \leq F(x_i) - F(x_{i-1}) \leq M_i\,(x_i - x_{i-1})\,.$$

It follows that

$$L(f,\mathcal{P}) \leq \sum_{i=1}^n [F(x_i) - F(x_{i-1})] \leq U(f,\mathcal{P})\,.$$

Finally, the claim follows from the above by observing that

$$F(b) - F(a) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})]\,.$$

□

Example 3: To compute

$$\int_0^1 x^2 \, dx \, ,$$

we use the second version of the fundamental theorem of calculus. Namely, observe that the derivative of the function

$$F(x) = \frac{x^3}{3}$$

is x^2 and therefore, by the second version of the fundamental theorem of calculus we have that

$$\int_0^1 x^2 \, dx = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3} \, .$$

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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