SPRING 2025 - 18.100B/18.1002

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Lecture 19

Question: What kind of functions are integrable?

Theorem: Any continuous function on $[a,b]$ is in $\mathcal{R}([a,b])$.

Basic properties of integrals.

Theorem: We have the following basic formulas for integrals:

(1) If $f \in \mathcal{R}([a,b])$ and $c \in \mathbf{R}$, then $c\,f \in \mathcal{R}([a,b])$ and
$$\int_a^b (c\,f)\,dx = c\int_a^b f\,dx\,.$$

(2) If $f,\, g \in \mathcal{R}([a,b])$, then $f+g \in \mathcal{R}([a,b])$ and
$$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx\,.$$

(3) If $f,\, g \in \mathcal{R}([a,b])$ and $f \leq g$, then
$$\int_a^b f\,dx \leq \int_a^b g\,dx\,.$$

(4) If $f \in \mathcal{R}([a,b])$ and $c \in (a,b)$, then $f \in \mathcal{R}([a,c])$ and $f \in \mathcal{R}([c,b])$ and
$$\int_a^c f\,dx + \int_c^b f\,dx = \int_a^b f\,dx\,.$$

Corollary: Suppose that $f,\, |f| \in \mathcal{R}([a,b])$, then
$$\int_x^b f\,dx\ \leq \int_a^b |f|\,dx\,.$$

Fundamental theorem of calculus, version 1: Let f be a continuous function on $[a,b]$ and define F on $[a,b]$ by

$$F(x)=\int_a^x f(s)\,ds\,.$$

The function F is differentiable with derivative f.

Fundamental theorem of calculus, version 2: Suppose that $F:[a,b]\to\mathbf{R}$ is differentiable and that $F'=f\in\mathcal{R}([a,b])$, then

$$F(b)-F(a)=\int_a^b f(s)\,ds\,.$$

Application of integrals: arclength.

Suppose that f and $g:[a,b]\to\mathbf{R}$ are differentiable functions and their derivatives are continuous, then we define the arclength of the curve

$$s\to(f(s),g(s))$$

by

$$L=\int_a^b\sqrt{(f'(s))^2+(g'(s))^2}\,ds\,.$$

Example 1: Suppose that $f(s)=s$ and $g(s)=s^2$, then $f'=1$ and $s'=2\,s$. Therefore, the arclength of the curve (s,s^2), where $s\in[0,1]$ is

$$L=\int_0^1\sqrt{1+(2\,s)^2}\,ds=\int_0^1\sqrt{1+4\,s^2}\,ds\,.$$

Improper integrals.

Unbounded interval.

Suppose that $f\in\mathcal{R}([a,b])$ for all $b>a$. If

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

exists, then we say that the improper integral

$$\int_a^\infty f(x)\,dx$$

exists and that

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

Example 2: On $[1,\infty)$, set

$$f(x) = \frac{1}{x^2}\,,$$

then

$$\int_1^c \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_c^1 = -\frac{1}{c}+1\,.$$

Since $-\frac{1}{c}+1\to 1$ as $c\to\infty$, the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx$$

exist and is equation to 1.

Example 3: On $[1,\infty)$, set

$$f(x) = \frac{1}{x}\,,$$

then

$$\int_1^c \frac{1}{x}\,dx = [\log x]_1^c = \log c\,.$$

The improper integral

$$\int_1^\infty \frac{1}{x}\,dx$$

does not exist.

Unbounded function.

Suppose that $f\in\mathcal{R}([c,b])$ for all $c>a$. If

$$\lim_{c\to a}\int_c^b f(x)\,dx$$

exists, then we say that the improper integral

$$\int_a^b f(x)\,dx$$

exists and that

$$\int_a^b f(x)\,dx = \lim_{c\to a}\int_c^b f(x)\,dx$$

Example 4: On $(0,1]$, set

$$f(x) = \frac{1}{\sqrt{x}}\,,$$

then

$$\int_c^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_c^1 = 2 - 2\sqrt{c}\,.$$

Since $2 - 2\sqrt{c} \to 2$ as $c \to 0$, the improper integral exists and is equal to

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\,.$$

Example 5: On $(0,1]$, set

$$f(x) = \frac{1}{x}\,,$$

then

$$\int_c^1 \frac{1}{x}\,dx = [\log x]_c^1 = -\log c\,.$$

Note that $-\log c \to \infty$ as $c \to 0$ so the improper integral does not exist.

Question: How do we define angle?

Answer: We define it through arclength.

On the unit circle

$$\{(x,y)\,|\,x^2+y^2=1\}$$

we define angle and the arclength. That is, suppose that (x,y) lies on the unit circle. The angle θ between $(1,0)$ and (x,y) is the arclength of the part of the unit circle from $(1,0)$ to (x,y). This part of the circle is parametrized by $(f(s),g(s)) = (s,\sqrt{1-s^2})$ and where $x \leq s \leq 1$. Since $f'(s) = 1$ and $g'(s) = -\frac{s}{\sqrt{1-s^2}}$ we get that

$$\theta = \int_x^1 \sqrt{1+\frac{s^2}{1-s^2}}\,ds = \int_x^1 \frac{1}{\sqrt{1-s^2}}\,ds\,.$$

The function $\arcsin x$ is defined by

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-s^2}}\, ds\,.$$

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
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