SPRING 2025 - 18.100B/18.1002

TOBIAS HOLCK COLDING

Lecture 20

Application of integrals: arclength.

Suppose that f and $g : [a, b] \to \mathbf{R}$ are differentiable functions and their derivatives are continuous, then we define the arclength of the curve

$$s \to (f(s), g(s))$$

by

$$L = \int_a^b \sqrt{(f'(s))^2 + (g'(s))^2}\, ds\,.$$

Example 1: Suppose that $f(s) = s$ and $g(s) = s^2$, then $f' = 1$ and $s' = 2\,s$. Therefore, the arclength of the curve (s, s^2), where $s \in [0, 1]$ is

$$L = \int_0^1 \sqrt{1 + (2\,s)^2}\, ds = \int_0^1 \sqrt{1 + 4\,s^2}\, ds\,.$$

Question: How do we define angle?

Answer: We define it through arclength.

On the unit circle

$$\{(x, y)\,|\, x^2 + y^2 = 1\}$$

we define angle and the arclength. That is, suppose that (x, y) lies on the unit circle. The angle θ between $(1, 0)$ and (x, y) is the arclength of the part of the unit circle from $(1, 0)$ to (x, y). This part of the circle is parametrized by $(f(s), g(s)) = (s, \sqrt{1 - s^2})$ and where $x \leq s \leq 1$. Since $f'(s) = 1$ and $g'(s) = -\frac{s}{\sqrt{1-s^2}}$ we get that

$$\theta = \int_x^1 \sqrt{1 + \frac{s^2}{1 - s^2}}\, ds = \int_x^1 \frac{1}{\sqrt{1 - s^2}}\, ds\,.$$

The function $\arccos x$ is defined by

$$\arccos x = \int_x^1 \frac{1}{\sqrt{1-s^2}}\, ds\,.$$

By the fundamental theorem of calculus we see that

$$\arccos x = -\frac{1}{\sqrt{1-x^2}}\,.$$

Pointwise convergence: Suppose that f_n is a sequence of functions on an interval I, then we say that f_n convergences pointwise to a function f if for all x we have

$$f_n(x) \to f(x)\,.$$

Example 1: Suppose that $f_n(x) = x^n$ on $[0,1]$, then f_n converges pointwise to f where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1\,. \end{cases}$$

Suppose first that $0 \leq x < 1$, then $f_n(x) = x^n \to 0$. If $x = 1$, then $f_n(x) = 1$ for all n and so $f_n(x) \to 1$. This show the claim.

Example 2: If $E_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$, then $E_n(x) \to \exp x$ pointwise. We have already proven that the radius of convergence for the power series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is infinity. From this the claim follows.

Uniform convergence: Suppose that f_n is a sequence of functions on an interval I, then we say that f_n convergences uniformly to a function f if for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then for all x

$$|f(x) - f_n(x)| < \epsilon\,.$$

Lemma 1: Suppose that I is an interval and f_n is a sequence of functions on I that converges uniformly to a function f, then f_n also converges pointwise to f.

Proof. This is immediate from the definition of uniform convergence. □

Example 1A: Suppose again that $f_n(x) = x^n$ on $[0,1]$, then f_n converges pointwise but NOT uniformly to f where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \,. \end{cases}$$

To see this observe that for each n,since f_n is continuous by the intermediate value theorem there exists x_n with $0 < x_n < 1$ such that $f_n(x) = \frac{1}{2}$. It now follows that

$$\frac{1}{2} = |f(x_n) - f_n(x_n)| \leq \sup_{x\in[0,1]} |f(x) - f_n| \,.$$

Thus we see that the convergence is not uniform. We already saw in Example 1 that the convergence is pointwise.

Example 2A: If $E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$, then $E_n(x) \to \exp x$ uniformly on any interval of the form $[-L, L]$. This will be a consequence of of Weirstrass M-test that we will discuss next.

Lemma 2 [Weirstrass M-test]: Suppose that I is an interval and f_n is a sequence of functions on I. Suppose also that M_n is a sequence of non-negative numbers with

$$|f_n(x)| \leq M_n \text{ for all } x \in I \,.$$

If the series

$$\sum_{n=1}^{\infty} M_n$$

converges, then the sequence of functions

$$S_n(x) = \sum_{k=0}^{n} f_k(x)$$

converges uniformly.

Proof. For each fixed x we have that that the sequence

$$\sum_{k=0}^{\infty} f_k(x)$$

converges. Moreover, we have that for all x and $m < n$ we have

$$|S_n(x) - S_m(x)| \leq |f_n(x)| + |f_{n-1}(x)| + \cdots + |f_{m+1}(x)| \leq M_n + \cdots + M_{m+1} \,.$$

For m fixed and since $S_n(x) \to S(x)$ it follows that

$$|S(x) - S_m(x)| \leq \sum_{k=m+1}^{\infty} M_k \,.$$

Since $\sum_{k=0}^{\infty} M_k$ is convergent it implies that given $\epsilon > 0$, there exists N such that if $m \geq N$, then $\sum_{k=m+1}^{\infty} M_k < \epsilon$. Therefore, for $m \geq N$ and all x

$$|S(x) - S_m(x)| < \epsilon \,.$$

This proves the claim. □

Example 2A: On the interval $I = [-L, L]$ suppose

$$f_n = \frac{x^n}{n!} \,.$$

Then

$$|f_n| \leq \frac{L^n}{n!} \,.$$

Since

$$\sum_n \frac{L^n}{n!}$$

is convergent Weirstrass M-test gives that the series

$$\sum_{n=0}^{\infty} f_n$$

converges uniformly on I.

Theorem: If

$$\sum_{k=0}^{\infty} a_k \, x^k$$

is a power series and R is its radius of convergence. Then it converges uniformly on any (finite) interval of the form $[-L, L]$ where $L < R$.

Proof. Recall that if $M = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}$, then the radius of convergence is $R = \frac{1}{M}$. It follows that if $|x| \leq L < R$, then

$$\limsup |a_n \, x^n|^{\frac{1}{n}} = |x| \, \limsup |a_n|^{\frac{1}{n}} \leq L \, M < 1 \,.$$

Choose $1 > \alpha > L \, M$. For n sufficiently large $|a_n \, x^n| \leq M_n = \alpha^n$. Since the geometric series $\sum_n \alpha^n$ is convergent, Weirstrass M-test gives the claim. □

Example 3: The geometric power series

$$\sum_{k=0}^{\infty} x^k$$

converges uniformly to $\frac{1}{1-x}$ on all intervals of the form $[-L, L]$ where $L < 1$. Since the radius of convergence of the power series is one the claim therefore follows from the theorem above.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
(screen-optimized)
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Portrait.pdf
(print-optimized)

MIT, Dept. of Math., 77 Massachusetts Avenue, Cambridge, MA 02139-4307.