SPRING 2025 - 18.100B/18.1002

TOBIAS HOLCK COLDING

Lecture 21

Theorem 1: Suppose that I is an interval and f_n is a sequence of continuous functions on I. If f_n converges uniformly to f, then f is also continuous.

Proof. Let x_0 in I be arbitrary but fixed. We will show that f is continuous at x_0. Given $\epsilon > 0$, since $f_n \to f$ uniformly, there exists a N such that if $n \geq N$, then for all x in I

$$|f(x) - f_n(x)| < \frac{\epsilon}{3}\,.$$

Since f_N is continuous at x_0, there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}\,.$$

Combining this gives that for $|x - x_0| < \delta$

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\,.$$

This gives the claim. □

Example 1: Set

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}\,,$$

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,.$$

In the previous lecture we showed that Weirstrass M-test implies that $E_n \to E$ uniformly on $[-L, L]$. Since each E_n is continuous we have from Theorem 1 that E is continuous.

Here is another useful way of thinking of uniform convergence. Recall that on the space of continuous functions $C(I)$ on an interval $I = [a, b]$ there is a natural metric given by that

$$d(f, g) = \max_{x \in I} \{|f(x) - g(x)| \,|\, x \in I\}\,.$$

We have the following:

Proposition: Let I be an interval $[a,b]$ and f_n, $f \in C(I)$, then $f_n \to f$ in the metric space if and only if f_n converges to f uniformly.

Proof. To see this note that

$$|f(x) - f_n(x)| \leq \epsilon \text{ for all } x \in I$$

if and only if

$$d(f,f_n) \leq \epsilon \,.$$

To say that $f_n \to f$ uniformly is therefore equivalent to that $d(f, f_n) \to 0$ giving the claim. □

From this we get:

Corollary: $C([a,b])$ is Cauchy complete.

Proof. Suppose that f_n is a Cauchy sequence in $C([a,b])$ we need to find a $f \in C([a,b])$ such that $f_n \to f$ uniformly. For each x fixed, the sequence $f_n(x)$ is a Cauchy sequence in $\mathbf{R}$. This follows since

$$|f_n(x) - f_m(x)| \leq d(f_n, f_m) \,.$$

Therefore, since $\mathbf{R}$ is Cauchy complete, for each x there exists a $f(x)$ such that $f_n(x) \to f(x)$. This defines the function f and show that $f_n \to f$ converges pointwise. We need to show that the convergence is uniform. To see that observe that given $\epsilon > 0$ since f_n is a Cauchy sequence, there exists N such that if n and $m \geq N$, then

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \text{ for all } x \in I \,.$$

Therefore, for $f(x) = \lim_{m\to\infty} f_m(x)$ we have

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon \text{ for all } x \in I \,.$$

This show that the convergence is uniform. □

Theorem 2: If $f_n \in \mathcal{R}([a,b])$ and $f_n \to f$ uniformly, then $f \in \mathcal{R}([a,b])$ and

$$\int_a^b f_n \, dx \to \int_a^b f \, dx \,.$$

Proof. We need to first show that $f \in \mathcal{R}([a,b])$ and so we need to show that given $\epsilon > 0$, there exists a partition $\mathcal{P}$ of the interval $[a,b]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon \,.$$

Since $f_n \to f$ uniformly we have that there exists a N such that if $n \geq N$, then

$$|f(x) - f_n(x)| < \frac{\epsilon}{3\,(b-a)}\,.$$

We have therefore that for any partition $\mathcal{P}$ that

$$|m_i^{f_n} \leq m_i^f| \leq \frac{\epsilon}{3\,(b-a)}\,,$$

$$|M_i^{f_n} \leq M_i^f| \leq \frac{\epsilon}{3\,(b-a)}\,,$$

It follows that for any partition when $n \geq N$, then

$$|U(f,\mathcal{P}) - U(f_n,\mathcal{P})| < \frac{\epsilon}{3}\,,$$

$$|L(f,\mathcal{P}) - L(f_n,\mathcal{P})| < \frac{\epsilon}{3}\,.$$

We can now use that since $f_N \in \mathcal{R}\,([a,b])$ we have that there exists a partition $\mathcal{P}$ such that

$$U(f_N,\mathcal{P}) - L(f_N,\mathcal{P}) < \frac{\epsilon}{3}\,.$$

Combining it all gives that

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &< U(f,\mathcal{P}) - U(f_N,\mathcal{P}) + U(f_N,\mathcal{P}) - L(f_N,\mathcal{P}) + L(f_N,\mathcal{P}) - L(f,\mathcal{P}) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\,. \end{aligned}$$

This show that $f \in \mathcal{R}\,([a,b])$. We also need to see that

$$\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b f_n\,dx\,.$$

This, however, follows from that

$$L(f,\mathcal{P}) \leq \int_a^b f\,dx \leq U(f,\mathcal{P})\,,$$

$$L(f_n,\mathcal{P}) \leq \int_a^b f_n\,dx \leq U(f_n,\mathcal{P})\,.$$

and that for $n \geq N$

$$|U(f,\mathcal{P}) - U(f_n,\mathcal{P})| < \frac{\epsilon}{3}\,,$$

$$|L(f,\mathcal{P}) - L(f_n,\mathcal{P})| < \frac{\epsilon}{3}\,.$$

Namely, we now have that also

$$L(f,\mathcal{P}) - \frac{\epsilon}{3} \leq \int_a^b f_n\,dx \leq U(f,\mathcal{P}) + \frac{\epsilon}{3}$$

and therefore

$$\int_a^b f\,dx - \int_a^b f\,dx \;< \epsilon\,.$$

□

Example 1A: Set

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}\,,$$

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,.$$

Then we have from Example 1 that $E_n \to E$ uniformly on $[-L, L]$. We now have from Theorem 2 that

$$\sum_{k=0}^{n} \int_0^1 \frac{x^k}{k!}\,dx \to \int_0^1 E(x)\,dx\,.$$

Theorem 3: Suppose that f_n are differentiable functions on $[a, b]$ and $x_0 \in [a, b]$. If

- $f_n(x_0) \to c$,
- $f_n' \to g$ uniformly,
- f_n' are continuous on $[a, b]$,

then there exists a differentiable function f with

- $f_n \to f$ uniformly,
- $f_n' \to f'$ uniformly.

Proof. Define a function F on $[a, b]$ by

$$f(x) = c + \int_{x_0}^{x} g\,dx\,,$$

and note since f_n' are continuous and that $f_n' \to g$ uniformly, it follows from Theorem 1 that g is also continuous. Therefore, by the fundamental theorem of calculus f is differentiable and $f' = g$. Moreover, by the fundamental theorem of calculus we have that

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'\,dx\,.$$

We are done provided we can show that $f_n \to f$. To see that note that

$$\begin{aligned} |f(x) - f_n(x)| &= \; c + \int_{x_0}^{x} g\,dx - f_n(x_0) - \int_{x_0}^{x} f_n'\,dx \\ \leq |c - f_n(x_0)| + \; \int_{x_0}^{x} (g - f_n')\,dx \; &\leq |c - f_n(x_0)| + \int_{x_0}^{x} |g - f_n'|\,dx \\ &\leq |c - f_n(x_0)| + (b-a)\,d(g, f_n')\,. \end{aligned}$$

The claim now follows since $f_n(x_0) \to c$ and $d(g, f_n') \to 0$. □

Example 1B: Set

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}\,,$$

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,.$$

Then

$$E_n' = \sum_{k=1}^{n} k\,\frac{x^{k-1}}{k!} = \sum_{k=0}^{n-1} \frac{x^{k-1}}{(k-1)!} = E_{n-1}\,.$$

From Example 1 that $E_{n-1} \to E$ uniformly on $[-L, L]$ and each E_n are continuous. Moreover, for all n we have that

$$E_n(0) = 1 = E(0)\,.$$

It follows therefore from Theorem 3 that

$$E_n' = E_{n-1} \to E'$$

uniformly and since $E_n' = E_{n-1}$, then we have that $E' = E$.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
TBB can be downloaded at:
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Landscape.pdf
(screen-optimized)
https://classicalrealanalysis.info/com/documents/TBB-AllChapters-Portrait.pdf
(print-optimized)

MIT, Dept. of Math., 77 Massachusetts Avenue, Cambridge, MA 02139-4307.