SPRING 2025 - 18.100B/18.1002

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Lecture 22

Suppose that a_n is a sequence and

$$\sum_{n=0}^{\infty} a_n\, x^n\,,$$

is a power series, the radius of convergence R is

$$R = \frac{1}{M} \text{ where } M = \limsup |a_n|^{\frac{1}{n}}\,.$$

Lemma: The radius of convergence is the same for the power series

$$\sum_{n=0}^{\infty} a_n\, x^n$$

as the power series

$$\sum_{n=1}^{\infty} n\, a_n\, x^{n-1}\,.$$

.

Proof. Since

$$n^{\frac{1}{n-1}} = \mathrm{e}^{\frac{\log n}{n-1}} \to 1\,,$$

and

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n-1}}$$

we have that

$$\limsup_{n\to\infty} |n\, a_n|^{\frac{1}{n-1}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\,.$$

From this the claim follows. □

Iterating this gives:

Corollary: The power series

$$\sum_{n=0}^{\infty} a_n\, x^n$$

has the same radius of convergence as the power series

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n\, x^{n-k}\,.$$

We now get the following:

Theorem: Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n\, x^n\,,$$

is a power series with radius of convergence R, then

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n\, x^{n-k}$$

and

$$\int f(x)\, dx = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}\, x^n\,.$$

Proof. Let us first argue for $= 1$. We will see that this is a consequence of Theorem 3 from Lecture 21. Set

$$f_n(x) = \sum_{k=0}^{n} a_k\, x^k$$

and

$$f(x) = \sum_{k=0}^{\infty} a_k\, x^k\,.$$

Moreover, let R be the radius of convergence for the power series f. We have the following three properties

(1)

$$f_n(0) = a_0 = f(0)\,.$$

(2) On each interval $[-L, L]$, where $L < R$, we have uniform convergence

$$f_n' \to \sum_{k=1}^{\infty} k\, a_k\, x^{k-1}\,.$$

(3) Each f_n' is continuous.

We see that Theorem 3 applies and show that

$$f' = \sum_{k=1}^{\infty} k\, a_k\, x^{k-1}\,.$$

Iterating this gives the claim for all k. Finally, the claim about the integral

$$\int f(x)\, dx$$

follows easily from Theorem 2 from Lecture 21. □

Ordinary differential equations: A differential equation is an equation that involves an unknown function and its derivative.

Example: Here are some examples of differential equations

$$f'(x) = x\,.$$
$$f'(x) - f^2(x) = 0\,.$$
$$f(x)\, f'(x)\, f''(x) = 1\,.$$

For the first of these and each constant c, the function

$$f_c(x) = \frac{1}{2}\, x^2 + c$$

is a solution. For the second

$$f(x) = \frac{1}{1-x}$$

is a solution. For the third $y = 0$ is a solution and so is $y = x$.

We will be interested in an ordinary differential equation (ODE) of the form

$$y' = f(y) + g(x)\,.$$

Here $y = y(x)$ is the unknown function and f, g are given functions. Note that while g only depend on x the function f also depend on the unknown function y.

We are interested in whether there exist solutions and when they exist if they are unique.

More precisely, suppose that we have the following:

- f be a continuously differentiable function on $\mathbf{R}$.
- g be a continuous function on $\mathbf{R}$.
- a is a real number.

We are intersted in existence and uniqueness of the ODE:

$$\begin{cases} y'(x) & = f(y(x)) + g(x) \\ y(0) & = a\,. \end{cases}$$

We will show next time the following:

Picard-Lindelöf theorem: There exists $\delta > 0$ such that there is a unique solution to this ODE on $(-\delta, \delta)$.

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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