SPRING 2025 - 18.100B/18.1002

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Lecture 23

Ordinary differential equations: A differential equation is an equation that involves an unknown function and its derivative.

Suppose that we have the following:

- f be a continuously differentiable function on $\mathbf{R}$.
- g be a continuous function on $\mathbf{R}$.
- a is a real number.

We will be intersted in existence and uniqueness of the ODE:

$$(\dagger) \qquad \begin{cases} y'(x) & = f(y(x)) + g(x)\,, \\ y(0) & = a\,. \end{cases}$$

We say that this is a first order equation since it only involves the function and its derivative and not higher derivatives.

The following theorem gives a satisfying answer to the question of existence and uniqueness for this ODE:

Picard-Lindelöf theorem: There exists $\delta > 0$ such that there is a unique solution to $(\dagger)$ on $(-\delta, \delta)$.

Before we prove this theorem let us recall a result that we have proven earlier. Suppose that $[a, b]$ is an interval and let $C([a, b])$ be the space of continuous functions on $[a, b]$. We equip this space with the metric d given by that if h_1, $h_2 \in C([a, b])$, then

$$d(h_1, h_2) = \max_{x \in [a,b]} |h_1(x) - h_2(x)|\,.$$

We proved earlier the following theorem:

Theorem 1: The metric space $(C([a,b]),d)$ is Cauchy complete.

We will also need to recall what it means for a map from a metric space to itself is contracting. A map T is a contracting map on a metric space (X,d) if for some $c<1$ and all x, $y\in X$

$$d(T(x),T(y))\leq c\,d(x,y)\,.$$

We shall also use that we have proven the following fact:

Theorem 2: If (X,d) is a Cauchy complete metric space and $T:X\to X$ is a contracting map, then T has a unique fix point.

Indeed this theorem was proven by showing that for any $x\in X$, the sequence x, $T(x)$, $T^2(x)$, $T^3(x),\cdots$ is a Cauchy sequence and the limit is the unique fix point of T. The proof of this used that

$$d(T^{n+1}(x),T^n(x))\leq c^n\,d(T(x),x)\,,$$

and therefore by the triangle inequality

$$d(T^{n+k}(x),T^n(x))\leq\sum_{i=1}^{k}d(T^{i+n}(x),T^{i-1+n}(x))\leq c^{i-1+n}\,d(T(x),x)\,.$$

Which is easily seen to imply that the sequence $T^n(x)$ is a Cauchy sequence.

We will also use the following lemma:

Lemma 1: Suppose that u_1 and u_2 are continuous functions on an interval I. Assume also that

- $u_1(x_0)=u_2(x_0)$.
- If $u_1(x)=u_2(x)$, then $u_1=u_2$ in a neighborhood of x.

then $u_1=u_2$.

Proof. Let

$$J_+=\{z\in I\,|\,z\geq x_0\text{ and }u_1(x)=u_2(x)\text{ for all }x\in[x_0,z]\}\,.$$

Then $x_0\in J_+$ so $J_+\neq\emptyset$. Let $z_0=\sup J_+$, if $z_0\in I$, then $u_1(z_0)=u_2(z_0)$ by continuity. Since also u_1 and u_2 agrees in a neighborhood of z_0 it follows that z_0 must be the right end point of I. Similarly one can show that $u_1=u_2$ everywhere to the left of z_0. This proves the lemma. □

Finally, in the proof of the Picard-Lindelöf theorem we will also need the next lemma. In this lemma f is a function on $\mathbf{R}$ as above, so f differentiable and the derivative of f is continuous and g will be a continuous function on $\mathbf{R}$. For $\delta > 0$, on the space of continuous functions on $[-\delta, \delta]$ we define a map T on functions y as follows

$$T(y)(x) = a + \int_0^x [f(z(s)) + g(s)]\, ds\,.$$

Note that when y is continuous, then so is $T(y)$.

Lemma 2: Let a be a constant and set $R = |a| + 2$. There exists a $\delta > 0$ such that:

- The map T maps the ball (in the metric space $(C([-\delta, \delta]), d)$) of radius R and with center the constant function zero into itself. We write $B_R(0)$ for this ball and so have that $T : B_R(0) \to B_R(0)$.
- The map T is contracting on $B_R(0)$.

Proof. Let

$$L_1 = \max_{|z| \leq R} |f(z)|\,,$$
$$L_2 = \max_{|x| \leq 1} |g(x)|\,.$$

We will first show that if that if we choose $\delta_0 > 0$ small enough, then T maps $B_R(0)$ into itself. That is, we will show that if $|y| \leq R$ on $[-\delta_0, \delta_0]$, then

$$|T(y)| \leq R\,.$$

To see this set

$$\delta_0 = \min\left\{1, \frac{1}{L_1 + 1}, \frac{1}{L_2 + 1}\right\}\,.$$

Now suppose that $|y| \leq R$ and $|x| \leq \delta_0$, then

$$|T(y)(x)| \leq |a| + \int_0^x |f(y(s))|\, ds + \int_0^x |g(s)|\, ds$$
$$\leq |a| + \delta_0\, L_1 + \delta_0\, L_2 \leq |a| + 2 = R\,.$$

This show that T maps $B_R(0)$ into itself.

Next set

$$M = \max_{|z| \leq R} |f'(z)|\,,$$

and

$$\delta = \min\left\{\delta_0, \frac{1}{2\,M + 1}\right\}\,.$$

Suppose that y_1 and y_2 are two continuous functions on $[-\delta, \delta]$ in $B_R(0)$, then

$$|T(y_1)(x) - T(y_2)(x)| = \left|\int_0^x [f(y_1(s)) - f(y_2(s))]\, ds\right|\,.$$

By the mean value theorem applied to f for each s we have a z_s between $y_1(s)$ and $y_2(s)$ such that

$$f(y_1(s)) - f(y_2(s)) = f'(z_s)\,(y_1(s) - y_2(s))\,.$$

Since $|y_i| \leq R$ we have that we have that for each s

$$|f(y_1(s)) - f(y_2(s))| \leq M\,|y_1(s) - y_2(s)| \leq M\ \max\,|y_1 - y_2| = M\,d(y_1, y_2)\,,$$

and therefore

$$|T(y_1)(x) - T(y_2)(x)| = \int_0^x [f(y_1(s)) - f(y_2(s))]\,ds\ \leq M\ \delta\,d(y_1, y_2) < \frac{1}{2}\,d(y_1, y_2)\,.$$

□

We are now ready to show the Picard-Lindelöf theorem:

Proof. (of the Picard-Lindelöf theorem.) Let T be defined as above and R and $\delta > 0$ be given by Lemma 2. A fixed point for T is a function y such that $T(y) = y$. By the fundamental theorem of calculus if y is a fix point of T, then we have that

$$y'(x) = (T(y))'(x) = f(y(x)) + g(x)\,.$$

Moreover, $y(0) = a$. In other words any fix point of T is a solution to the ODE.

We need to show that the solution is unique. Suppose that y^* is any other solution, then by the fundamental theorem of calculus

$$y^*(x) = a + \int_0^x (y^*)'(s)\,ds = T(y^*(s))\,.$$

Note that this holds even if the interval I that y^* is defined on (containing 0) is different from $[-\delta, \delta]$. We have from this that any solution is a fix point of T. Since T is contracting on $B_R(0)$ it follows that for any fix point with $|y| \leq R$, then y is unique. In general, suppose that y_1 and y_2 are two solutions defined on intervals I_1 and I_2 both containing 0. We have from the above that they agree in a neighborhood of 0. The argument in Lemma 2 that proved uniqueness in a small neigborhood of 0 works equally well in a neighborhood of any other point. It now follow from Lemma 1 that y_1 and y_2 agrees everywhere. □

References

[TBB] B.S. Thomson, J.B. Bruckner, and A.M. Bruckner, *Elementary Real Analysis, 2nd edition*
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