18.100B Spring 2025 Problem Set 1

Problem 1 (10pt)**.** Let $\mathbb{F}$ be an ordered field with $1 \neq 0$. Show that $1 > 0$. You can find the axioms of an ordered field on the next page. Hint: Show $(-1)(-1) = 1$ first.

solution 1. Since $1 \neq 0$, we know either $1 > 0$ or $0 > 1$ occurs from (O1)

Problem 2 (10pt)**.** Recall that two rational numbers $\frac{n_1}{m_1}$ and $\frac{n_2}{m_2}$ are the same, denoted by $\frac{n_1}{m_1} = \frac{n_2}{m_2}$, if $n_1 m_2 = n_2 m_1$. Show that the addition

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}$$

is well-defined. That is, if $\frac{n_1}{m_1} = \frac{n_2}{m_2}$ and $\frac{p_1}{q_1} = \frac{p_2}{q_2}$, then

$$\frac{n_1}{m_1} + \frac{p_1}{q_1} = \frac{n_2}{m_2} + \frac{p_2}{q_2}.$$

Problem 3 (10pt)**.** Find $\sup E$ and $\inf E$ for the following sets E:

(1) $E = \{n \in \mathbb{Z} \mid n < \sqrt{12}\}$
(2) $E = \{r \in \mathbb{Q} \mid r < \sqrt{12}\}$
(3) $E = \{x \in \mathbb{R} \mid x^2 - x - 1 < 0\}$
(4) $E = \{\frac{n^2+n}{n^2+1} \mid n \in \mathbb{N}\}$

No proof is needed for the answer.

Problem 4 (10pt)**.** Let $\mathbb{M}$ be the set of polynomials with integer coefficients.

$$\mathbb{M} := \{f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n \mid a_i \in \mathbb{Z}\}.$$

Define the relation $0 \prec f$ if $0 < f(x)$ for x large enough. To be more precise, we say $0 \prec f$ if there exists $M > 0$ such that $f(x) > 0$ for all $x > M$. Then define $f \prec g$ if $0 \prec g - f$. Show that $(\mathbb{M}, \prec)$ is an ordered set. You can find the axioms for an ordered set (O1) and (O2) on the next page.
You can use the following fact directly (without proving it):
If $f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$ with $0 < a_n$, then $0 < f(x)$ for x large enough.

Problem 5 (10pt)**.** Let $(\mathbb{M}, \prec)$ be the ordered set defined in Problem 4. Show that $(\mathbb{M}, \prec)$ **doesn't** satisfy the Archimedean property.

Problem 6 (10pt)**.** The greatest lower bound of a set E is defined to be the number β which has the following properties:

- For all $x \in E$, $\beta \leq x$.
- Suppose $\alpha \leq x$ for all $x \in E$. Then $\beta \geq \alpha$.

Show that for any non-empty set $E \subset \mathbb{R}$ which is bounded from below, E has the greatest lower bound.

Problem 7 (20pt)**.** Show that for all real number $x \in \mathbb{R}$, there exists a real number $y \in \mathbb{R}$ such that $y^3 = x$. Hint: start with the case $x > 0$.

An ordered field $\mathbb{F}$ is a set with two operations, addition $+$ and multiplication $\cdot$, and one relation $<$, which satisfy the following axioms:

(A) Axioms for addition

(A1) If $x \in \mathbb{F}$ and $y \in \mathbb{F}$, then $x + y \in \mathbb{F}$.

(A2) $x + y = y + x$ for all $x, y \in \mathbb{F}$.

(A3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}$.

(A4) There exists an element $0 \in \mathbb{F}$ such that $0 + x = x$ for all $x \in \mathbb{F}$.

(A5) For all $x \in \mathbb{F}$, there exists an element $-x \in \mathbb{F}$ such that $x + (-x) = 0$.

(M) Axioms for multiplication

(M1) If $x \in \mathbb{F}$ and $y \in \mathbb{F}$, then $xy \in \mathbb{F}$.

(M2) $xy = yx$ for all $x, y \in \mathbb{F}$.

(M3) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$.

(M4) There exists an element $1 \in \mathbb{F}$ such that $1 \cdot x = x$ for all $x \in \mathbb{F}$.

(M5) For all $x \in \mathbb{F}$ and $x \neq 0$, there exists an element $1/x \in \mathbb{F}$ such that $x \cdot (1/x) = 1$.

(D) Axiom of distribution $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$.

(O) Axioms of order

(O1) If $x \in \mathbb{F}$ and $y \in \mathbb{F}$, then one and only one of the statement

$$x < y, \ x = y, \ y < x$$

is true.

(O2) If $x < y$ and $y < z$, then $x < z$.

(OA) Axiom of order and addition If $y < z$, then $y + x < z + x$.

(OM) Axiom of order and multiplication If $0 < x$ and $0 < y$, then $0 < xy$.

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