18.100B Spring 2025 Problem Set 2

Problem 1 (25pt)**.** Let a_n and b_n be a sequence of real numbers.

(1) Assume that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. Show that $\lim_{n\to\infty}(a_nb_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$.
(2) Give an example in which $\lim_{n\to\infty}(a_nb_n)$ exists but neither $\lim_{n\to\infty} a_n$ nor $\lim_{n\to\infty} b_n$ exists.

Problem 2 (25pt)**.** Find the limit for the following sequence if it exists. Or show that the limit doesn't exist.

(1) $a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n}$
(2) $a_n = \frac{\sin(n)}{n}$
(3)

$$a_n = \underbrace{\frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+2}} + \frac{n^2}{\sqrt{n^6+3}} + \cdots + \frac{n^2}{\sqrt{n^6+n}}}_{n \text{ terms}}$$

Problem 3 (15pt)**.** Let a_n be a sequence of real numbers and L be a real number. Show that the following two statements are equivalent. One holds if and only if the other does.

- There exists a subsequence a_{n_k} converging to L.
- For any $\epsilon > 0$, there exist infinite any a_n in $(L-\epsilon, L+\epsilon)$.

Problem 4 (15pt)**.** Where possible find a subsequence that is monotone and a subsequence that is convergent for the following sequences.

(1) $a_n = \sin(n\pi/8)$
(2) $a_n = (-1)^n n$