Problem Set 3

Problem 1 (10pt)**.** Let $x < y$ be two real numbers. Show that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.
Hint: Use the Archimedean property. You can also use the following fact. Suppose S is a non-empty subset of $\mathbb{Z}$ which is bounded from above. Then S has a maximum element.

Problem 2 (20pt)**.** Let E be an non-empty subset of $\mathbb{R}$ which is bounded. Define

$$F := \{x^2 \mid x \in E\}.$$

Show that $\sup F$ exists and that $\sup F = \max\{(\sup E)^2, (\inf E)^2\}$.

Problem 3 (10pt)**.** Let E be an non-empty subset of $\mathbb{R}$ which is bounded from above. Show that there is a sequence a_n such that $a_n \in E$ and $\lim_{n\to\infty} a_n = \sup E$.
Hint: Show that for all $\epsilon > 0$, there exists $a \in E$ such that $a > \sup E - \epsilon$.

Problem 4 (15pt)**.** Let $a_1 = 4$ and define a_n inductively by

$$a_n = 4 - \frac{4}{a_{n-1}} \text{ for } n \geq 2.$$

Show that $\lim_{n\to\infty} a_n = 2$.
Hint: Show that $a_n \geq 2$ and that a_n is monotone decreasing.

Problem 5 (20pt)**.** Let $T : \mathbb{R} \to \mathbb{R}$ be a contraction map and $x \in \mathbb{R}$ be a number. Define a sequence a_n by requiring $a_1 = x$ and $a_{n+1} = T(a_n)$.

(1) Show that for any $m \in \mathbb{N}$, $|a_1 - a_m| \leq \frac{1}{1-\lambda}|a_1 - a_2|$
(2) Show that a_n is a Cauchy sequence.

You can find the definitions of a contraction map and of a Cauchy sequence on the next page.

Definition 1. A map $T : \mathbb{R} \to \mathbb{R}$ is called a contraction map if there exists $\lambda \in (0, 1)$ such that

$$|T(x) - T(y)| \leq \lambda |x - y| \text{ for all } x, y \in \mathbb{R}.$$

Definition 2. A sequence a_n of real numbers is called a Cauchy sequence if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x_m| \leq \epsilon$ for all $n, m \geq N$.

Theorem. *Every Cauchy sequence of real numbers converges.*