# Problem Set 4

**Problem 1** (10pt)**.** Give an example of a sequence $a_n$ that satisfies the following two conditions.

- $a_n$ is divergent.
- For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$ for all $n \geq N$.

**Problem 2** (10pt)**.** Let $p > 0$ be a positive number. Consider the $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

In the lecture we proved that the series diverges for $p = 1$ and converges for $p = 2$. Show that the series converges for $p > 1$ and diverges for $0 < p \leq 1$.

**Problem 3** (15pt)**.** Let $r > 0$ be a positive number. Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges for the following cases.

(1) $a_n = \sqrt{n+r} - \sqrt{n}$
(2) $a_n = n^3 r^n$
(3) $a_n = \frac{1}{n!} r^n$

The answer may depend on the value of $r$.

**Problem 4** (20pt)**.** Let $b_n$ be a sequence of non-negative numbers which decreases to zero. That is,

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0 \text{ and } \lim_{n\to\infty} b_n = 0.$$

Let $a_n = (-1)^{n-1} b_n$. The purpose of this problem is to show that $\sum_{n=1}^{\infty} a_n$ converges. This is called the *alternating series test.* Let

$$s_n = \sum_{k=1}^{n} a_k.$$

(1) Show that $s_{2k+1}$ is decreasing and that $s_{2k}$ is increasing.
(2) Show that $s_{2k+1}$ is bounded from below and that $s_{2k}$ is bounded from above.
(3) Show that both $\lim_{k\to\infty} s_{2k+1}$ and $\lim_{k\to\infty} s_{2k}$ exist and are identical.
(4) Show that $\sum_{n=1}^{\infty} a_n$ converges.

Problem 5 is on the next page

**Problem 5** (20pt)**.** For any real number $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the largest integer which is less or equal to $x$. Equivalently, $\lfloor x \rfloor$ is the integer such that

$$0 \leq x - \lfloor x \rfloor < 1.$$

For example $\lfloor 1.3 \rfloor = 1$, $\lfloor -3 \rfloor = -3$ and $\lfloor \sqrt{5} \rfloor = 2$. Define a sequence

$$a_n = \sqrt{2}n - \lfloor \sqrt{2}n \rfloor.$$

(1) Show that $a_n$ has a convergent subsequence.
(2) Let $N \in \mathbb{N}$ be an integer. Suppose $0 < a_m < 1/N$ for some integer $m$. Show that there exists $n \in \mathbb{N}$ such that

$$a_n > 1 - \frac{1}{N}.$$

Hint: Consider $n = km$ for integers $k$.

(3) Show that for all $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$a_n > 1 - \frac{1}{N}$$

Hint: Applying the following fact to $a_n$. Suppose $a_1, a_2, \ldots a_{N+1}$ are $N + 1$ numbers contained in $[0, 1)$. Then there exist two distinct numbers $k$ and $\ell$ such that $|a_k - a_\ell| < \frac{1}{N}$.

(4) Let $E := \{a_n \mid n \in \mathbb{N}\}$. Show that there exists a sequence $b_k \in E$ such that

$$\lim_{k \to \infty} b_k = 1.$$

We remark that $b_k$ is **not** required to be a subsequence of $a_n$ and that $b_k$ are **not** required to be distinct between different $k$'s.

The purpose of (2), (3) and (4) in Problem 5 is to prepare ingredients that eventually show that $a_n$ has a subsequence which converges to 1. The last piece of ingredient is the following lemma.

**Lemma 1.** Let $a_n$ be a sequence of real numbers and $E := \{a_n \mid n \in \mathbb{N}\}$. Suppose there exists a sequence $b_k \in E$ such that

$$\lim_{k \to \infty} b_k = L.$$

Further assume that $L \notin E$. Then $a_n$ has a subsequence $a_{n_k}$ such that

$$\lim_{k \to \infty} a_{n_k} = L.$$

The proof can be found on the next page.

*Proof.* Take $n_1 = 17$. (17 is arbitrary) Set $\epsilon_1 > 0$ be the minimum among $1, |L - a_1|, |L - a_2|, \dots |L - a_{n_1}|$. Here we used $L \notin E$ to ensure $\epsilon_1 > 0$. Because

$$\lim_{k \to \infty} b_k = L,$$

there exists $n_2 \in \mathbb{N}$ such that $|a_{n_2} - L| < \epsilon_1$. Since for all $j = 1, 2, \dots, n_1$,

$$|a_{n_2} - L| < \epsilon_1 \leq |a_j - L|,$$

we must have $n_2 > n_1$.
Set $\epsilon_2 > 0$ be the minimum among $1/2, |L - a_1|, |L - a_2|, \dots |L - a_{n_2}|$. Here we used $L \in E$ to ensure $\epsilon_2 > 0$. Because

$$\lim_{k \to \infty} b_k = L,$$

there exists $n_3 \in \mathbb{N}$ such that $|a_{n_3} - L| < \epsilon_2$. Since for all $j = 1, 2, \dots, n_2$,

$$|a_{n_3} - L| < \epsilon_2 \leq |a_j - L|,$$

we must have $n_3 > n_2$. Inductively we obtain $n_1 < n_2 < n_3 < \dots$ and

$$|a_{n_k} - L| < \epsilon_k \leq \frac{1}{k} \to 0$$

. This finishes the proof. $\square$