Problem Set 5

Problem 1 (10pt)**.** Let a_n, b_n be two sequences of real numbers and x be a real number. The *Fourier series* is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Suppose that $\sum_{n=1}^{\infty}(|a_n| + |b_n|)$ converges, show that the above Fourier series converges for all $x \in \mathbb{R}$.

Problem 2 (20pt)**.**

(1) Give an example of a function $f(x)$ defined on $[-1, 1]$ with the following property: $(f(x))^2$ is continuous on $[-1, 1]$ but $f(x)$ is **not** continuous on $[-1, 1]$.
(2) Suppose that $f(x)$ is a function defined on an interval I and that $(f(x))^3$ is continuous on I. Show that $f(x)$ is also continuous on I.

Hint: In problem 2 of Pset 2, we showed that the function $g(x) := x^{1/3}$ is defined. Show that this is a continuous function. You can use the fact that $x < y \Rightarrow x^3 < y^3$ without proof.

Problem 3 (10pt)**.** Recall that for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer which is less or equal to x. Equivalently, $\lfloor x \rfloor$ is the integer such that

$$\lfloor x \rfloor \leq x < 1 + \lfloor x \rfloor.$$

Determine at which points the function $f(x) = \lfloor x \rfloor$ is continuous or discontinuous.

Problem 4 (10pt)**.** Let $f(x)$ and $g(x)$ be two continuous functions defined on $\mathbb{R}$ with $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

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Problem 5 (20pt)**.** Recall that

$$E(x) := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

In Pset 4 and the lecture, we showed the series is convergent for all $x \in \mathbb{R}$ and hence $E(x)$ is defined for all $x \in \mathbb{R}$.

(1) For $k \in \mathbb{N}$, define

$$E_k(x) := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} = \sum_{n=0}^{k} \frac{x^n}{n!}.$$

Show that for any $k \in \mathbb{N}$, $E_k(x)$ is a continuous function on $\mathbb{R}$.

Hint: Use the fact that x^k are continuous.

(2) Let $M > 0$ be a fixed number. Show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|E_k(x) - E(x)| < \epsilon$$

for all $k \geq N$ and for all $x \in [-M, M]$.

Remark: We require that a **single** number N works **for all** $x \in [-M, M]$.
Hint: Show that $|E_k(M) - E(M)| \geq |E_k(x) - E(x)|$.

(3) Show that $E(x)$ is continuous on $\mathbb{R}$.

Hint: To show $E(x)$ is continuous at x_0, consider

$$|E(x) - E(x_0)| \leq |E(x) - E_k(x)| + |E_k(x) - E_k(x_0)| + |E_k(x_0) - E(x_0)|.$$

Then apply the results in part (1) and part (2).

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