Problem Set 6

Problem 1 (10pt)**.** Let $X = \{\text{continuous functions defined on } [0,1]\}$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d(f,g) = \max_{x\in[0,1]} \ f(x) - g(x) \ .$$

Show that d satisfies the triangle inequality. In other words, for all continuous functions $f(x), g(x)$ and $h(x)$

$$d(f,h) \leq d(f,g) + d(g,h).$$

Problem 2 (10pt)**.** Let $X = \mathbb{N} = \{1,2,3,\dots\}$ and $d : X \times X \to \mathbb{R}$ defined by

$$d(n,m) = \ \frac{1}{n} - \frac{1}{m} \ .$$

(1) Show that the sequence $x_n = 5n$ is a Cauchy sequence.
(2) Show that the sequence $x_n = 5n$ doesn't converge.

As a result, this metric space is not Cauchy complete.

Problem 3 (20pt)**.** In this problem we consider the metric space with $X = \mathbb{R}^2$ and

$$d\Big((x_1,y_1),(x_2,y_2)\Big) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

(1) Let (x_n, y_n) be a sequence in $\mathbb{R}^2$. Show that (x_n, y_n) converges if and only if both x_n and y_n converge as sequences of $\mathbb{R}$.
(2) Let (x_n, y_n) be a sequence in $\mathbb{R}^2$. Show that (x_n, y_n) is a Cauchy sequence if and only if both x_n and y_n are Cauchy sequences of $\mathbb{R}$.
(3) Show that $\mathbb{R}^2$ is Cauchy complete.

Problem 4 (10pt)**.** Let $A_j, j \in \mathbb{N}$ be open sets in a metric space (X,d).

(1) Show that $\displaystyle\bigcup_{j=1}^{\infty} A_j$ is open.
(2) Show that $\displaystyle\bigcap_{j=1}^{\infty} A_j$ may not be open. Hint: Consider $X = \mathbb{R}$ and $A_j = (0, 1 + \frac{1}{j})$.

Problem 5 is on the next page

Problem 5 (20pt)**.** Let (X,d) be a metric space, K be a compact set in X and $A_j, j \in \mathbb{N}$ be closed sets in X. Suppose for all $k \in \mathbb{N}$,

$$K \cap \left(\bigcap_{j=1}^{k} A_j \right) \text{ is non-empty.}$$

Show that

$$K \cap \left(\bigcap_{j=1}^{\infty} A_j \right) \text{ is non-empty.}$$

Hint: Consider open sets $B_j = A_j^c$. Express $\bigcap_{j=1}^{\infty} A_j$ and $\bigcap_{j=1}^{k} A_j$ in terms of B_j. Then use the definition of a compact set.

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