Problem Set 7

Problem 1 (20pt)**.** Consider the set

$$X := \left\{ (a_1, a_2, a_3, \dots) \quad \sum_{j=1}^{\infty} a_j^2 \text{ converges} \right\}.$$

Define the function $d : X \times X \to \mathbb{R}$ as the following. For $x = (a_1, a_2, a_3, \dots)$ and $y = (b_1, b_2, b_3, \dots)$ in X,

$$d(x, y) := \sqrt{\sum_{j=1}^{\infty} (a_j - b_j)^2}.$$

(1) Show that the function d is well-defined. Equivalently, show that for $x = (a_1, a_2, a_3, \dots)$ and $y = (b_1, b_2, b_3, \dots)$ in X,

$$\sum_{j=1}^{\infty} (a_j - b_j)^2 \text{ converges.}$$

(2) Show that the function d satisfies the triangle inequality. You can use the following triangle inequality in $\mathbb{R}^n$ without proof. For all $n \geq 1$,

$$\sqrt{\sum_{j=1}^{n} (a_j - c_j)^2} \leq \sqrt{\sum_{j=1}^{n} (a_j - b_j)^2} + \sqrt{\sum_{j=1}^{n} (b_j - c_j)^2}.$$

(3) Consider a sequence in X as $x_1 = (1, 0, 0, 0, \dots)$, $x_2 = (0, 1, 0, 0, \dots)$, and so on. In general, $x_n = (0, \dots, 0, \overbrace{1}^{\text{nth}}, 0, \dots)$. Show that x_n has no convergent subsequence.

Problem 2 (10pt)**.** Let (X, d) be a metric space and x_n be a sequence in X. Denote $E = \{x_1, x_2, x_3, \dots\}$. Suppose x_n has no convergent subsequence. Show that for all $k \in \mathbb{N}$, there exists $r_k > 0$ such that

$$B(x_k, r_k) \cap E = \{x_k\}.$$

You can use the following fact without proof. Fix $x \in X$. Suppose for all $r > 0$, there are infinite many elements in $E \cap B(x, r)$. Then x_n has a subsequence which converges to x.

Remark: You might find it easier to first show this when $(X, d) = \mathbb{R}$ and then adapt your argument to a general metric space (X, d).

Problem 3 (10pt)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $f(x)$ is differentiable on $(-\infty, 0) \cup (0, \infty)$ and that $\lim_{x \to 0} f'(x)$ exits. Show that $f(x)$ is differentiable at $x = 0$ and that $f'(0) = \lim_{x \to 0} f'(x)$.

Hint: See the Mean Value Theorem on the last page.

Problem 4 (10pt)**.** Use $\frac{d}{dx} e^x = e^x$ to show that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.
Hint: Apply the Mean Value Theorem with x and 0 being the endpoints.

Problem 5 (20pt)**.** Let $X = \{\text{continuous functions defined on } [0,1]\}$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d(f, g) = \max_{x \in [0,1]} \; f(x) - g(x) \; .$$

Suppose $f_n \in X$ is a Cauchy sequence.

(1) Fix an arbitrary $x_0 \in [0,1]$. Show that $\lim_{n \to \infty} f_n(x_0)$ exists.

(2) Define the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) := \lim_{n \to \infty} f_n(x).$$

Show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$f_n(x) - f(x) \;\; \leq \epsilon$$

for all $x \in [0,1]$ and for all $n \geq N$.

(3) Show that $f(x)$ is continuous on $[0,1]$. Equivalently, show that $f \in X$.
Hint: To show $f(x)$ is continuous at x_0, consider

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

(4) Show that $\lim_{n \to \infty} f_n = f$ as a sequence in X.

Theorem. (Mean Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be a function. Suppose $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b). Then there exists a number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$