Problem Set 8

Problem 1 (10pt). Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Suppose that both f'(x) and f''(x) are continuous functions on \mathbb{R} and that f(0) = 0. Define the function

$$g(x) := \begin{cases} f(x)/x & x \neq 0, \\ f'(0) & x = 0. \end{cases}$$

Show that q'(x) exists for all $x \in \mathbb{R}$ and express q'(x) in terms of f(x) and its derivatives.

Problem 2 (15pt). Suppose there exist two functions $S: \mathbb{R} \to \mathbb{R}$ and $C: \mathbb{R} \to \mathbb{R}$ which satisfy the following properties:

- $\begin{array}{l} \bullet \ \, \frac{d}{dx}S(x) = C(x), \ \, \frac{d}{dx}C(x) = S(x). \\ \bullet \ \, S(0) = 0, \, C(0) = 1. \end{array}$
- (1) Let $S^{(n)}(x)$ be the nth derivative of S(x). Show that for $k \in \mathbb{N} \cup \{0\}$,

$$S^{(2k)}(x) = S(x), S^{(2k+1)}(x) = C(x).$$

(2) Show that for all $x \in \mathbb{R}$,

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Problem 3 (20pt). Let a < b be two real numbers and $f: [a,b] \to \mathbb{R}$ be a function. Suppose f(x) is continuous on [a,b] and differentiable on (a,b). Suppose f'(x)>0 for all $x \in (a,b)$.

- (1) Show that f(x) is strictly increasing on [a, b]. That is, $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in [a,b].
- (2) Show that for all $y \in (f(a), f(b))$, there exists a unique $x \in (a, b)$ such that f(x) = y.
- (3) Let the function $g:(f(a),f(b))\to(a,b)$ be the inverse function of f(x). In other words, g(y) = x if f(x) = y. Show that g is continuous on (f(a), f(b)).
- (4) Show that g is differentiable on (f(a), f(b)) and that

$$g'(y) = \frac{1}{f'(g(y))}$$
 for all $y \in (f(a), f(b))$.

Problem 4 (10pt). Define the function $f: [-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in [-1,0) \cup (0,1], \\ 0 & x = 0 \end{cases}$$

Show that f(x) is Riemann integrable on [-1,1] and that $\int_{-1}^{1} f(x)dx = 2$.

Problem 5 (15pt). Let $f:(a,b)\to\mathbb{R}$ be a function. f(x) is called **convex** if for all $x_1< x_2\in (a,b)$ and all $\lambda\in [0,1]$,

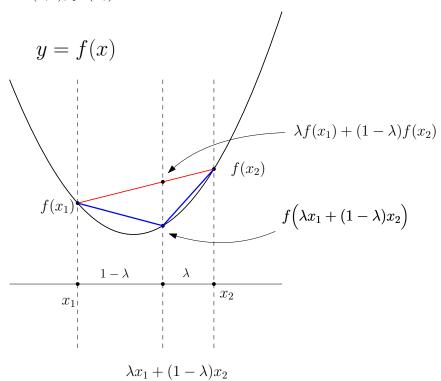
$$f(\lambda x_1 + (1 - \lambda)x_2) \le \lambda f(x_1) + (1 - \lambda)f(x_2).$$

- (1) Suppose that f''(x) exits and is non-negative for all $x \in (a, b)$. Show that f(x) is convex.
 - Hint: See the picture below and apply the Mean Value Theorem.
- (2) Suppose that f''(x) exits for all $x \in (a, b)$. Show that

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

Hint: Use the L'Hopital's rule.

(3) Suppose that f''(x) exits for all $x \in (a, b)$ and that f(x) is convex. Show that for all $x \in (a, b)$, $f''(x_0) \ge 0$.



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