Problem Set 8

Problem 1 (10pt)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that both $f'(x)$ and $f''(x)$ are continuous functions on $\mathbb{R}$ and that $f(0) = 0$. Define the function

$$g(x) := \begin{cases} f(x)/x & x \neq 0, \\ f'(0) & x = 0. \end{cases}$$

Show that $g'(x)$ exists for all $x \in \mathbb{R}$ and express $g'(x)$ in terms of $f(x)$ and its derivatives.

Problem 2 (15pt)**.** Suppose there exist two functions $S : \mathbb{R} \to \mathbb{R}$ and $C : \mathbb{R} \to \mathbb{R}$ which satisfy the following properties:

- $\frac{d}{dx}S(x) = C(x)$, $\frac{d}{dx}C(x) = S(x)$.
- $S(0) = 0$, $C(0) = 1$.

(1) Let $S^{(n)}(x)$ be the nth derivative of $S(x)$. Show that for $k \in \mathbb{N} \cup \{0\}$,

$$S^{(2k)}(x) = S(x),\ S^{(2k+1)}(x) = C(x).$$

(2) Show that for all $x \in \mathbb{R}$,

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Problem 3 (20pt)**.** Let $a < b$ be two real numbers and $f : [a, b] \to \mathbb{R}$ be a function. Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose $f'(x) > 0$ for all $x \in (a, b)$.

(1) Show that $f(x)$ is strictly increasing on $[a, b]$. That is, $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in $[a, b]$.
(2) Show that for all $y \in (f(a), f(b))$, there exists a unique $x \in (a, b)$ such that $f(x) = y$.
(3) Let the function $g : (f(a), f(b)) \to (a, b)$ be the inverse function of $f(x)$. In other words, $g(y) = x$ if $f(x) = y$. Show that g is continuous on $(f(a), f(b))$.
(4) Show that g is differentiable on $(f(a), f(b))$ and that

$$g'(y) = \frac{1}{f'(g(y))} \quad \text{for all } y \in (f(a), f(b)).$$

Problem 4 (10pt)**.** Define the function $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in [-1, 0) \cup (0, 1], \\ 0 & x = 0 \end{cases}$$

Show that $f(x)$ is Riemann integrable on $[-1, 1]$ and that $\displaystyle\int_{-1}^{1} f(x)dx = 2$.

Problem 5 (15pt)**.** Let $f : (a,b) \to \mathbb{R}$ be a function. $f(x)$ is called **convex** if for all $x_1 < x_2 \in (a,b)$ and all $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

(1) Suppose that $f''(x)$ exits and is non-negative for all $x \in (a,b)$. Show that $f(x)$ is convex.
Hint: See the picture below and apply the Mean Value Theorem.

(2) Suppose that $f''(x)$ exits for all $x \in (a,b)$. Show that

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

Hint: Use the L'Hopital's rule.

(3) Suppose that $f''(x)$ exits for all $x \in (a,b)$ and that $f(x)$ is convex. Show that for all $x \in (a,b)$, $f''(x_0) \geq 0$.