Problem Set 9

Problem 1 (15pt)**.** Let $f(x)$ and $g(x)$ be functions defined on $[a,b]$. Suppose that $f(x)$ and $g(x)$ are Riemann integrable and that $f(x) \leq g(x)$ for all $x \in [a,b]$.

(1) Show that

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Hint: Consider $\displaystyle\int_a^b [g(x) - f(x)]\,dx$.

(2) Further assume $f(x)$ and $g(x)$ are continuous and

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

Show that $f(x) = g(x)$ for all $x \in [a,b]$.

Problem 2 (15pt)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(1) Show that for any $[a,b] \subset \mathbb{R}$, there exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(2) Assume $f(x)$ is differentiable on $(0,1)$ and there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (0,1)$. Show that for all $n \in \mathbb{N}$,

$$\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{j}{n}\right) \leq \frac{M}{n}$$

Problem 3 (10pt)**.** Let $F(x)$ and $G(x)$ be two differentiable functions defined on $[a,b]$. Further assume that $F'(x)$ and $G'(x)$ are continuous on $[a,b]$. Show that

$$\int_a^b F'(x)\,G(x)\,dx = F(b)\,G(b) - F(a)\,G(a) - \int_a^b F(x)\,G'(x)\,dx.$$

Problem 4 (10pt)**.** Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $\varphi : [A,B] \to \mathbb{R}$ be a differentiable function with $\varphi'(x) > 0$. Further assume that $\varphi(A) = a$, $\varphi(B) = b$ and that $\varphi'(x)$ is continuous on $[A,B]$. Show that

$$\int_a^b f(x)\,dx = \int_A^B f(\varphi(y))\,\varphi'(y)\,dy.$$

Problem 5 (20pt)**.** In this problem, we concern the uniqueness of the continuous solution of the equation

$$\begin{aligned} y'(x) &= y(x)^2, \text{ for all } x \in [0,1], \\ y(0) &= a. \end{aligned} \tag{$\star$}$$

(1) Let $y(x)$ be a continuous function on $[0,1]$. Show that $y(x)$ solves $(\star)$ if and only if

$$y(x) = a + \int_0^x y(t)^2\,dt \text{ for all } x \in [0,1]. \tag{$\star\star$}$$

(2) Let $y_1(x)$ and $y_2(x)$ be two continuous functions on $[0,1]$. Show that there exists $1 > b > 0$ such that the following holds. Suppose $y_1(x)$ and $y_2(x)$ both satisfy $(\star\star)$ and $y_1(x_0) = y_2(x_0)$ for some $x_0 \in [0, 1-b]$. Then

$$y_1(x) = y_2(x) \text{ for all } x \in [x_0, x_0 + b].$$

Hint: Pick a suitable b to deduce

$$\max_{x \in [x_0, x_0+b]} |y_1(x) - y_2(x)| \le \frac{1}{2} \max_{x \in [x_0, x_0+b]} |y_1(x) - y_2(x)|.$$

(3) Let $y_1(x)$ and $y_2(x)$ be two continuous functions on $[0,1]$. Suppose $y_1(x)$ and $y_2(x)$ both solve $(\star)$. Show that $y_1(x) = y_2(x)$ for all $x \in [0,1]$.

18.100B Real Analysis
Spring 2025