Problem Set 10

Problem 1 (15pt)**.** Show that

$$\phi(x) = \sum_{n=0}^{\infty} \frac{x^{2\,n}}{n!}\,, \tag{0.1}$$

is a solution of the differential equation $\phi' = 2\,x\,\phi$ on $\mathbb{R}$ without first finding an explicit formula for ϕ.

Problem 2 (15pt)**.** Show that if $g_n \to g$ uniformly on $[a, b]$ and each g_n is continuous, then the sequence of functions

$$Fn(x) = \int_a^x g_n(s)\,ds$$

also converges uniformly on $[a, b]$.

Problem 3 (15pt)**.** Show that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\,,$$

converges uniformly on $[-L, L]$ for every $L \in \mathbb{R}$, but does not converge uniformly on all of the real line. (Does it converge pointwise on the real line?) Obtain a series representation for

$$\int_{-L}^{L} \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.$$

Problem 4 (15pt)**.** Show that the equation

$$x^2\,y^2 + 2\,e^{x\,y} - 4 - 2\,e^2 = 0$$

can be solved for y in terms of x in a neighborhood of the point $x = 1$ with $y(1) = 2$. Calculate y' when $x = 1$.

Problem 5 (15pt)**.** Let f_n be a sequence of Lipschitz functions on $[a, b]$ with common Lipschitz constant L. (This means that

$$|f_n(x) - f_n(y)| \le L\,|x - y|$$

for all $n \in \mathbb{N}$, $x, y \in [a, b]$.)

(1) If $f = \lim_n f_n$ pointwise, then f is continuous and, in fact, satisfies a Lipschitz condition with constant L.
(2) If $f = \lim_n f_n$ pointwise the convergence is uniform.

(3) Show by example that the results in (1) and (2) fail if we weaken our hypotheses by requiring only that each function is a Lipschitz function. (Here, the constant L may depend on n.)