SPRING 2025 - 18.100B/18.1002

TOBIAS HOLCK COLDING

Review for the Final

Definition (Closed subsets): Let (X, d) be a metric space. We say that C is a closed subset of X if the complement $X \setminus C$ is open.

Note that $\emptyset$ (the empty set) and X are both closed.

Lemma: Let (X, d) be a metric space and $r > 0$, then

$$A_r = \{y \,|\, d(x, y) > r\}$$

is open. Equivalently, $\bar{B}_r(x) = \{y \,|\, d(x, y) \leq r\}$ is closed.

Theorem: A subset C of a metric space (X, d) is closed if and only if for all convergent sequences x_n with all x_n in C also the limit is in C.

Theorem:

- Union: If C_α is a family of closed subsets, then $\cap_\alpha C_\alpha$ is also closed.
- Intersection: If $C_1, \cdots, C_n$ are closed subsets, then $C_1 \cup \cdots \cup C_n$ is also closed.

Warning: Union of infinitely many closed sets may not be closed!!!

Definition (Cover, open cover and finite sub-cover): If A is a subset of X, then a cover of A is a collection collection of subsets U_α of X so that

$$A \subset \cup_\alpha U_\alpha \,.$$

We say that a $U_{\alpha_1}, \cdots, U_{\alpha_n}$ is a finite sub-cover if also $\{U_{\alpha_i}\}_i$ is a cover.

If (X, d) is a metric space and all the U_α are open, then we say that $\{U_\alpha\}_\alpha$ is an open cover.

Definition (Compact subset): If (X, d) is a metric space and A is a subset, then we say that A is compact if each open cover has a finite sub-cover.

Theorem: (Heine-Borel.) $[a, b]^n \subset \mathbf{R}^n$ is compact.

Theorem: If (X, d) is a metric space and A a compact subset, then A is closed and bounded.

Warning: The converse is not the case!!! There are closed a bounded subsets of metric spaces that are not compact.

Theorem: If (X, d) is a metric space and A a compact subset, then any closed subset C contained in A is also compact.

Theorem: If (X, d) is a metric space and A a compact subset, then any sequence in A has a convergent subsequence.

Definition: If $f : \mathbf{R} \to \mathbf{R}$ is a function, then we say that f is differentiable at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. (Note that in this fraction x is assumed to be $\neq x_0$.) When the limit exists, then we say that the function f is differentiable at x_0 and that its derivative at x_0 is the limit. In this case we denote the derivative at x_0 by $f'(x_0)$.

Lemma: If f is differentiable at x_0, then f is continuous at x_0.

Theorem: If f, g are functions on $\mathbf{R}$ that both are differentiable at x_0, then

- (Sum rule.)
$$(f + g)'(x_0) = f'(x_0) + g'(x_0)\,.$$
- (Leibniz's rule.)
$$(f\, g)(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)\,.$$
- (Quotient rule.) If also $g(x_0) \neq 0$, then
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{g^2(x_0)}\,.$$

Theorem: (Chain rule.) If $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbf{R}$ are functions, where f is differentiable at x_0 and g differentiable at $y_0 = f(x_0)$, then the composition $g \circ f$ is differentiable at x_0 and the derivative at x_0 is

$$(g \circ f)'(x_0) = g'(y_0)\, f'(x_0)\,.$$

Lemma: Let $f : [a, b] \to \mathbf{R}$ be a differentiable function and suppose that $a < x_0 < b$ and that f has a local maximum or minimum at x_0, then

$$f'(x_0) = 0\,.$$

Theorem: (Mean value theorem.) Let $f : [a, b] \to \mathbf{R}$ be a differentiable function, then there exists a x_0 between a and b such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}\,.$$

.

Theorem: (Cauchy mean value theorem.) Let $f,\, g : [a, b] \to \mathbf{R}$ be differentiable functions, then there exists a x_0 between a and b such that

$$f'(x_0)\,[g(b) - g(a)] = g'(x_0)\,[f(b) - f(a)]\,.$$

In particular, if $g(b) - g(a) \neq 0$, then

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}\,.$$

Theorem: (L'Hopital's rule, version 1.) Let $f,\, g : (a, b) \to \mathbf{R}$ be differentiable functions with $g(x) \neq 0$ and $g'(x) \neq 0$ for all x, assume that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0\,.$$

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}\,.$$

Theorem: (L'Hopital's rule, version 2.) Let $f,\, g : (a, b) \to \mathbf{R}$ be differentiable functions with $g(x) \neq 0$ and $g'(x) \neq 0$ for all x, assume that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty\,.$$

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}\,.$$

Theorem: (Taylor expansion.) Let $f : [a,b] \to \mathbf{R}$ be a function and k a positive integer. Assume that f, $f', f^{(2)}, \cdots, f^{(k-1)}$ exists on $[a,b]$ and are continuous and that $f^{(k)}$ is defined on (a,b), then there exists c between a and b such that

$$f(b) = f(a) + f'(a)\,(b-a) + \frac{f^{(2)}(a)}{2}\,(b-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}\,(b-a)^{k-1}$$
$$+\frac{f^{(k)}(c)}{(k)!}\,(b-a)^k\,.$$

For and infinitely differentiable function f on $\mathbf{R}$ we define the $(k-1)$ Taylor polynomial at a by

$$P_{k-1}(x) = f(a) + f'(a)\,(x-a) + \frac{f^{(2)}(a)}{2}\,(x-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}\,(x-a)^{k-1}\,.$$

Riemann integrals

Partition: Let $[a,b]$ be an interval. A partition $\mathcal{P}$ of the interval $[a,b]$ is a number of sub-divisions x_i such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b\,.$$

The partition is then the sub-intervals $[x_{i-1}, x_i]$. We will set $\Delta\, x_i = x_i - x_{i-1}$.

Upper and lower sums: Suppose now that $f : [a,b] \to \mathbf{R}$ is a bounded function and that $\mathcal{P} = \{x_i\}$ is a partition of the interval $[a,b]$. We define upper and lower sums as follows. Set

$$M_i = \sup_{[x_{i-1},x_i]} f\,,$$
$$m_i = \inf_{[x_{i-1},x_i]} f\,,$$

and upper $U(f,\mathcal{P})$ and lower sums $L(f,\mathcal{P})$ by

$$U(f,\mathcal{P}) = \sum_{i=1}^{n} M_i\;\Delta\, x_i\,,$$
$$L(f,\mathcal{P}) = \sum_{i=1}^{n} m_i\;\Delta\, x_i\,.$$

Upper and lower integrals: Suppose now that $f : [a,b] \to \mathbf{R}$ is a bounded function. Define the upper Riemann integral of f by

$$\overline{\int_a^b} f\,dx = \inf_{\mathcal{P}} U(f,\mathcal{P})\,.$$

Here the infimum is taken over all partitions of $[a, b]$. Likewise, we define the lower Riemann integral by

$$\underline{\int_a^b} f\, dx = \sup_{\mathcal{P}} L(f, \mathcal{P})\,.$$

Riemann integral: Suppose that $f : [a, b] \to \mathbf{R}$ is a bounded function, then we say that f is Riemann integrable if

$$\overline{\int_a^b} f\, dx = \underline{\int_a^b} f\, dx\,.$$

If the function is Riemann integrable, then the Riemann integral is

$$\int_a^b f\, dx = \overline{\int_a^b} f\, dx = \underline{\int_a^b} f\, dx\,.$$

The Riemann integrable functions is denoted by $\mathcal{R}([a, b])$.

Theorem: Any continuous function on $[a, b]$ is in $\mathcal{R}([a, b])$.

Definition: Uniformly continuous. Suppose that $f : I \to \mathbf{R}$ is a function, where I is an interval. We say that f is uniformly continuous if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ if } |x - y| < \delta\,.$$

Note that being uniformly continuous is stronger than being continuous. It means that for a given $\epsilon > 0$, the same δ can be used for all x.

Basic properties of integrals.

Theorem: We have the following basic formulas for integrals:

(1) If $f \in \mathcal{R}([a, b])$ and $c \in \mathbf{R}$, then $c\, f \in \mathcal{R}([a, b])$ and

$$\int_a^b (c\, f)\, dx = c \int_a^b f\, dx\,.$$

(2) If $f,\, g \in \mathcal{R}([a, b])$, then $f + g \in \mathcal{R}([a, b])$ and

$$\int_a^b (f + g)\, dx = \int_a^b f\, dx + \int_a^b g\, dx\,.$$

(3) If $f,\, g \in \mathcal{R}([a, b])$ and $f \le g$, then

$$\int_a^b f\, dx \le \int_a^b g\, dx\,.$$

(4) If $f \in \mathcal{R}([a,b])$ and $c \in (a,b)$, then $f \in \mathcal{R}([a,c])$ and $f \in \mathcal{R}([c,b])$ and

$$\int_a^c f\,dx + \int_c^b f\,dx = \int_a^b f\,dx\,.$$

Corollary: Suppose that f, $|f| \in \mathcal{R}([a,b])$, then

$$\int_x^b f\,dx \;\leq \int_a^b |f|\,dx\,.$$

Fundamental theorem of calculus, version 1: Let f be a continuous function on $[a,b]$ and define F on $[a,b]$ by

$$F(x) = \int_a^x f(s)\,ds\,.$$

The function F is differentiable with derivative f.

Fundamental theorem of calculus, version 2: Suppose that $F : [a,b] \to \mathbf{R}$ is differentiable and that $F' = f \in \mathcal{R}([a,b])$, then

$$F(b) - F(a) = \int_a^b f(s)\,ds\,.$$

Suppose that f and $g : [a,b] \to \mathbf{R}$ are differentiable functions and their derivatives are continuous, then we define the arclength of the curve

$$s \to (f(s), g(s))$$

by

$$L = \int_a^b \sqrt{(f'(s))^2 + (g'(s))^2}\,ds\,.$$

Improper integrals.

Unbounded interval.

Suppose that $f \in \mathcal{R}([a,b])$ for all $b > a$. If

$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$

exists, then we say that the improper integral

$$\int_a^\infty f(x)\,dx$$

exists and that

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

Unbounded function.

Suppose that $f \in \mathcal{R}([c,b])$ for all $c > a$. If

$$\lim_{c\to a}\int_c^b f(x)\,dx$$

exists, then we say that the improper integral

$$\int_a^b f(x)\,dx$$

exists and that

$$\int_a^b f(x)\,dx = \lim_{c\to a}\int_c^b f(x)\,dx$$

Pointwise convergence: Suppose that f_n is a sequence of functions on an interval I, then we say that f_n convergences pointwise to a function f if for all x we have

$$f_n(x) \to f(x)\,.$$

Uniform convergence: Suppose that f_n is a sequence of functions on an interval I, then we say that f_n convergences uniformly to a function f if for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then for all x

$$|f(x) - f_n(x)| < \epsilon\,.$$

Lemma 1: Suppose that I is an interval and f_n is a sequence of functions on I that converges uniformly to a function f, then f_n also converges pointwise to f.

Lemma [Weirstrass M-test]: Suppose that I is an interval and f_n is a sequence of functions on I. Suppose also that M_n is a sequence of non-negative numbers with

$$|f_n(x)| \leq M_n \text{ for all } x \in I\,.$$

If the series

$$\sum_{n=1}^\infty M_n$$

converges, then the sequence of functions

$$S_n(x) = \sum_{k=0}^n f_k(x)$$

converges uniformly.

Theorem: If

$$\sum_{k=0}^{\infty} a_k \, x^k$$

is a power series and R is its radius of convergence. Then it converges uniformly on any (finite) interval of the form $[-L, L]$ where $L < R$.

Theorem: Suppose that I is an interval and f_n is a sequence of continuous functions on I. If f_n converges uniformly to f, then f is also continuous.

Proposition: Let I be an interval $[a, b]$ and f_n, $f \in C(I)$, then $f_n \to f$ in the metric space $(C(I), d)$ if and only if f_n converges to f uniformly.

Corollary: $C([a, b])$ is Cauchy complete.

Theorem: If $f_n \in \mathcal{R}\,([a, b])$ and $f_n \to f$ uniformly, then $f \in \mathcal{R}\,([a, b])$ and

$$\int_a^b f_n \, dx \to \int_a^b f \, dx \, .$$

Theorem: Suppose that f_n are differentiable functions on $[a, b]$ and $x_0 \in [a, b]$. If

- $f_n(x_0) \to c$,
- $f_n' \to g$ uniformly,
- f_n' are continuous on $[a, b]$,

then there exists a differentiable function f with

- $f_n \to f$ uniformly,
- $f_n' \to f'$ uniformly.

Suppose that a_n is a sequence and

$$\sum_{n=0}^{\infty} a_n \, x^n \, ,$$

is a power series, the radius of convergence R is

$$R = \frac{1}{M} \text{ where } M = \limsup |a_n|^{\frac{1}{n}} \, .$$

Corollary: The power series

$$\sum_{n=0}^{\infty} a_n \, x^n$$

has the same radius of convergence as the power series

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \, a_n \, x^{n-k} \, .$$

Theorem: Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n \, x^n \, ,$$

is a power series with radius of convergence R, then

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \, a_n \, x^{n-k}$$

and

$$\int f(x) \, dx = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} \, x^n \, .$$

Ordinary differential equations: Sppose that we have the following:

- f be a continuously differentiable function on $\mathbf{R}$.
- g be a continuous function on $\mathbf{R}$.
- a is a real number.

We are intersted in existence and uniqueness of the ODE:

$$\begin{cases} y'(x) & = f(y(x)) + g(x) \\ y(0) & = a \, . \end{cases}$$

Picard-Lindelöf theorem: There exists $\delta > 0$ such that there is a unique solution to this ODE on $(-\delta, \delta)$.

MIT, Dept. of Math., 77 Massachusetts Avenue, Cambridge, MA 02139-4307.
E-mail address: colding@math.mit.edu