SPRING 2025 - 18.100B/18.1002

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Review for Midterm

Least upper bound of a bounded set and **greatest lower bound** of a bounded set.

A sequence of real numbers is a function $f : \mathbf{N} \to \mathbf{R}$.

We usually use the notation $a_n = f(n)$.

Example 1: $\sqrt{2}$ is the limit of $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, $a_4 = 1.414$ etc.

Example 2: The sequence $a_n = \frac{1}{n}$ has zero as its limit.

Limit: Let a_n be a sequence and a a real number. We say that a_n converges to a if for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that if $n \geq N$, then

$$|a_n - a| < \epsilon\,.$$

If this is the case, then we also say that a is the limit of the sequence and we say that the sequence is convergenet.

A sequence that is not covergent is said to be divergent.

Theorem If a_n is a convergent sequence, then the set $\{a_n\}$ is a bounded subset of $\mathbf{R}$.

Basic algebraic properties of limits:

Theorem Suppose that a_n and b_n are convergenet sequences with $\lim a_n = a$, $\lim b_n = b$ and $C \in \mathbf{R}$, then

(1) $c_n = C\, a_n$ is convergenet and $\lim_{n\to\infty} c_n = C\, a$.
(2) $c_n = a_n + b_n$ is convergent and $\lim_{n\to\infty} c_n = a + b$.
(3) $c_n = a_n\, b_n$ is convergent with $\lim_{n\to\infty} c_n = a\, b$.
(4) If $b_n \neq 0$, $b \neq 0$ and $c_n = \frac{a_n}{b_n}$, then c_n is convergent and $\lim_{n\to\infty} c_n = \frac{a}{b}$.

Subsequence: Recall that a sequence a_n is a function $f : \mathbf{N} \to \mathbf{R}$ where we set $a_n = f(n)$. A subsequence b_n of a_n is a composition of functions $f \circ g$ where $g : \mathbf{N} \to \mathbf{N}$ is a strictly increasing function. So $b_n = f(g(n))$. Sometimes a subsequence of the sequence a_n also denoted by a_{n_k}.

Theorem: A sequence a_n is convergent with limit a if and only if all subsequences of a_n are also convergent with limit a.

Monotone convergence theorem:

- Increasing version. Let a_n be a monotone increasing sequence so with $a_n \leq a_{n+1}$. If the sequence is bounded from above, then a_n is convergent and
$$\lim_{n\to\infty} a_n = \sup\{a_n\}\,.$$
- Decreasing version. Let a_n be a monotone decreasing sequence so with $a_{n+1} \leq a_n$. If a_n is bounded from below, then a_n converges and
$$\lim_{n\to\infty} a_n = \inf\{a_n\}\,.$$

Cauchy sequence: A sequence a_n is said to be a Cauchy sequence if for all $\epsilon > 0$, there exists an N such that if m, $n \geq N$, then
$$|a_n - a_m| < \epsilon\,.$$
(Tail of the sequence bunch together.)

Definition A contracting map is a map $T : \mathbf{R} \to \mathbf{R}$ such that there exists $c < 1$ so for all x, $y \in \mathbf{R}$ we have that
$$|T(x) - T(y)| \leq c\,|x - y|\,.$$
(Points are squeezed together under the map.)

Theorem **(Cauchy convergence theorem)**: A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Definition A contracting map is a map $T : \mathbf{R} \to \mathbf{R}$ such that there exists $c < 1$ so for all $x,\, y \in \mathbf{R}$ we have that

$$|T(x) - T(y)| \leq c\,|x - y|\,.$$

(Points are squeezed together under the map.)

Contracting mapping theorem: Any contracting map has a fixed point.

Bolzano - Weirstrass theorem: Any bounded sequence of real numbers has a convergent subsequence.

A function $f : \mathbf{R} \to \mathbf{R}$ is said to be continuous at a point $x_0 \in \mathbf{R}$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon\,.$$

A function is said to be continuous if it is continuous at all points in the domain.

Theorem: If $f : \mathbf{R} \to \mathbf{R}$ is continuous and x_n is a sequence with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Algebraic properties of continuous functions:

- If f and g are continuous functions, then so is $f + g$.
- If f is continuous and c is a constant, then $c\,f$ is continuous.
- If f and g are continuous, then $f\,g$ is also continuous.
- If f is continuous and $f \neq 0$, then $\frac{1}{f}$ is continuous.
- If $f(x)$ and $g(x)$ are continuous, then $f(g(x))$ is continuous.

Theorem: All polynomials are continuous.

Extreme Value Theorem: Suppose that $f : [a, b] \to \mathbf{R}$ is a continuous function, then there exist $x_M \in [a, b]$ such that $f(x_M) \geq f(x)$ for all $x \in [a, b]$. Similarly, there exists $x_m \in [a, b]$ such that $f(x_m) \leq f(x)$ for all $x \in [a, b]$.

Intermediate Value Theorem: Suppose that $f : [a, b] \to \mathbf{R}$ is a continuous function, then for all y between $f(a)$ and $f(b)$, there exists $x \in [a, b]$ such that $f(x) = y$.

Series: Suppose that a_n is a sequence, we can form a new sequence s_n by setting

$$s_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i \,.$$

A series $\sum_{i=1}^{\infty} a_i$ converges if the sequence s_n converges and if it do we also write $\sum_{i=1}^{\infty} a_i$ for the limit.

Geometric series: Suppose now that $a_n = c^n$ so the series is

$$s_n = \sum_{i=0}^{n} c^i \,.$$

This is the geometric series. It is convergent precisely when $|c| < 1$. Moreover, when $|c| < 1$, then the limit (infinite sum) is

$$\sum_{i=0}^{\infty} c^i = \frac{1}{1-c} \,.$$

and diverges if $|c| \geq 1$.

Harmonic series: The series $\sum_{i=1}^{\infty} \frac{1}{2}$ is called the harmonic series. This series is divergent.

Absolutely convergent; We say that a series

$$\sum_{n=0}^{\infty} a_n$$

is absolutely convergent if the series

$$\sum_{n=0}^{\infty} |a_n|$$

is convergent. Absolutely convergent implies convergent but not the other way around.

Theorem: A series of non-negative numbers

$$\sum_{i=0}^{\infty} a_i \,,$$

where $a_n \geq 0$, is convergent if and only if the sequence s_n is bounded from above.

Example: The series

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

is convergent.

To help determine whether or not a series converges there are a number of tests:

- Comparison test.
- Ratio test.
- Root test.

Comparison test:

- **Version 1**: Suppose that a_n and b_n are two sequences with $0 \leq a_n \leq b_n$. If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.
- **Version 2**: Suppose that a_n and b_n are two sequences with $b_n \neq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, The series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\sum_{n=1}^{\infty} b_n$ is.

Ratio test: Let $a_n \geq 0$ and assume that $\frac{a_{n+1}}{a_n} \to a$. If

- $a < 1$, then the series $\sum a_n$ is convergent.
- $a > 1$, then the series $\sum a_n$ is divergent.
- $a = 1$, it is inconclusive.

Root test: Let $a_n \geq 0$ be a sequence of non-negative numbers. Suppose $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = r$. If

- $r < 1$, then the series $\sum_{n=0}^{\infty} a_n$ is convergent.
- $r > 1$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.
- $r = 1$, then it is inconclusive.

Power series: Let c_n be a sequence, then $\sum_{n=0}^{\infty} c_n\, x^n$ is a power series.

lim sup and lim inf of a sequence.
Let a_n be a sequence. If it is not bounded from above, then we set $\limsup a_n$ to be ∞. Otherwise we will define a new sequence b_n from a_n as follows.

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}\,.$$

Note that since we are assuming that the a_n's are bounded from above the b_n's are real numbers and the sequence b_n is decreasing. – It is decreasing since

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} \geq \sup\{a_{n+1}, a_{n+2}, \cdots\} = b_{n+1}\,.$$

(For b_{n+1} supremum is taken over a smaller set.) Since the sequence b_n is decreasing it is converging with limit b that possibly could be $-\infty$ if the sequence b_n is not bounded from below.

Definition (of lim sup):

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n = b\,.$$

Definition: (of radius of convergence). If $\sum_{n=0}^{\infty} a_n\, x^n$ is a power series. Set

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}\,.$$

R is said to be the radius of convergence.

Convention: If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$, then the radius of convergence is said to be ∞. If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$, then we set $R = 0$.

From the root test one can now show the following:

The power series in convergent if $|x| < R$ and divergent if $|x| > R$.

The case of where $|x| = R$ has to be examined on a case by case basis.

Exponential map as a power series: Define $E(x)$ as the power series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.$$

Definition: Metric space A metric space is a set X with a function $d : X \times X \to \mathbf{R}$ with the following three properties:

(1) $d(x, y) \geq 0$ for all x, $y \in X$ and $d(x, y) = 0$ if and only if $x = y$. (Distances ≥ 0.)

(2) $d(x,y) = d(y,x)$. (Symmetric.)
(3) $d(x,z) \leq d(x,y) + d(y,z)$. (Triangle inequality.)

Examples:

(1) $X = \mathbf{R}$ and
$$d(x,y) = |x-y|\,.$$
(2) $X = \mathbf{R}^2$ and for $x = (x_1, x_2)$ and $y = (y_1, y_2)$
$$d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}\,.$$
(3) $X = \mathbf{R}^3$ and for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$
$$d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2}\,.$$

Example: Continuous function on an interval $[a,b]$. Let $X = C([a,b])$ where $C([a,b])$ is the set of continuous functions on $[a,b]$. The distance between two continuous functions f and g is then
$$d(f,g) = \max_{x \in [a,b]} |f(x - g(x)|\,.$$

Sequences in a metric space: A sequence in a metric space (X,d) is a map $f : \mathbf{N} \to X$. We typically denote the image $f(n)$ by x_n. Similarly we define a **subsequence** as the composition of a strictly increasing map $g : \mathbf{N} \to \mathbf{N}$ with f and $x_{n_k} = f(g(k))$.

Definition: Convergent sequence in a metric space If (X,d) is a metric space and x_n is a sequence in X, then we say that x_n converges to x and write $x_n \to x$ or $x = \lim_{n\to\infty} x_n$ if for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then
$$d(x, x_n) < \epsilon\,.$$
This is equivalent to that the sequence $d(x_n, x_\infty) \to 0$.

Definition: Cauchy sequence in a metric space If (X,d) is a metric space and x_n is a sequence in X, then we say that x_n is a Cauchy sequence if for all $\epsilon > 0$, there exists an N, such that if m, $n \geq N$, then
$$d(x_m, x_n) < \epsilon\,.$$

Theorem: In any metric space (X,d) a convergent sequence is also a Cauchy sequence.

The converse is not always the case: If $X = (0,1) \subset \mathbf{R}$ with $d(x,y) = |x-y|$, then the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence but since 0 is not in X, it is not convergent. We sometimes express this by saying that in this case X is not Cauchy complete.

Examples of problems that you could be asked on the midterm:

For the midterm you are allowed one page of notes. However, the use of any outside material such as notes, calculators, and electronic devices is **not** allowed. Please show your work, and present your arguments in a **coherent, legible** manner. Unsupported and illegible arguments may not receive full credit.

Example 1: Show that the Box distance (from lecture 11) on $\mathbf{R}^2$ gives a metric space? What does it mean that a sequence is a Cauchy sequence in the Box metric?

Example 2: What is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} n^5 \, x^n \, ?$$

What is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^5 \, x^n}{n!} \, ?$$

Example 3: Suppose that a sequence is given recursively by $x_1 = a$ and

$$x_{n+1} = \frac{x_n}{n} \, .$$

For what a is the sequence convergent? When the sequence is convergent what is the limit?

Example 4: Suppose that

$$\sum_{n=1} a_n$$

is a series and

$$|a_n| < \frac{1}{n^2} \, .$$

Either show that the series is convergent or give an example of such a series that is divergent.

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