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PROFESSOR: All right, so last time we defined the integral of a non-negative measurable function, the Lesbesguen rule. Now we are going to define the Lesbesguen rule for a general class, a more general class of functions. So Lesbesgue integrable functions.

So what does this mean? Let E be a measurable subset of R, a function f, a measurable function f, from E to R. So its real value is Lesbesgue integrable over E. So I should say this is a measurable function-- is Lesbesgue integrable over E if the integral of the absolute value of f is finite. So a mark.

So recall that we have the positive and negative parts of f. So we can write f equals f plus minus f minus, where these are the positive and negative parts of f, both non-negative functions so that the absolute value of f is equal to f plus plus f minus. And therefore, if I want to compute the integral of the absolute value, this is equal to the integral of the positive part plus the negative part.

Remember, these are both non-negative functions. So let me just recall for you f plus is equal to max f0. f minus is equal to-- I think d for some reason, the f plus is always easy. The f minus is always-- max minus f0. So the integral of the absolute value is equal to the integral of f plus plus the integral of f minus.

These are both defined because these are both non-negative measurable functions. So both non-negative because this is a max of two things, one of them involving 0. So that's always bigger than or equal to 0.

So these are two non-negative measurable functions. These integrals exist. So this is finite if and only if both of these two things are finite. So thus f is-- instead of saying Lesbesgue integrable, I'm just going to say is integrable. This is equivalent to the functions f plus and f minus, the positive and negative parts of f, are integrable.

All right, so with that remark there, so if f is a measurable function from E to a measurable set, the R is integrable. So "intbl." My short form is usually-- you can sound it out. If f is Lesbesgue integrable, then the Lesbesgue integral of f over E is defined to be the integral of f plus over E minus the integral of f minus.

Now again, so this is a bit of new terminology. This is meaningful because I'm only defining the integral for integrable functions, meaning this is a finite number, this is a finite number. So I can always subtract two finite numbers. So this is the definition of being Lesbesgue integrable, the Lesbesgue integral.

So what are some immediate properties of integrable functions and the integral? Suppose f and g from E to R are integrable. Then for all c in R, the integral of c times f-- no, I should say c times f is integrable. And the integral of c times f is equal to c times the integral of f. That's one simple property that just follows from the definition along with the linearity that we have for the integral of non-negative functions over non-negative scalars.

f plus g is integrable. And the integral of f plus g over E is equal to the integral over E of f plus the integral of g. And again, just an analog of what we had for the integral of non-negative functions. If A and B are disjoint measurable sets, then integral of f over A union B is equal to the integral over A of f plus the integral over B of f.

One I'm just going to leave to you. It's pretty clear. I just write c as-- if it's 0, then this follows immediately. If c is positive, then it doesn't change the positive and negative parts. The positive part of c times f is just c times the positive part of f. And the negative part is just c times the negative part of f is the negative part of c times f. But it flips them if c is negative. So you can just check that equality for those two cases.

So let's move on to something more interesting, so for example, the fact that the integral is linear in f and g. So note that by the triangle inequality, I have that f plus g is less than or equal to f plus g, the absolute-- so the absolute value of f plus g is less than or equal to the absolute value of f plus the absolute value of g. And therefore, by what we know for the integral of non-negative measurable functions, the integral of f plus g is less than or equal to the integral of the absolute value of f plus the absolute value of g, which equals by linearity for non-negative measurable functions is equal to the sum of the integrals.

Each of these is finite. So therefore, their sum is finite. So if I have two integrable functions, their sum is integrable. Now why is the integral of the sum equal to the sum of the integrals? Well, we have f plus g, splitting f and g into their positive and negative parts and writing f as f plus minus f minus.

I can write f plus g as f plus plus g plus minus f minus plus g minus. Now this does not say that the positive part of f plus g is equal to this or that the negative part of f plus g is this. But it does say that the positive part of f plus g plus the sum of the negative parts is equal to-- just by splitting the left side into positive and negative parts, I get this identity.

Now all of these are non-negative measurable functions. Everything appearing here are non-negative measurable functions. And therefore, the integral, which is linear for non-negative measurable functions, tells me that the integral of f plus g plus the integral of f minus plus g minus equals the integral of f plus plus g plus over E plus the integral over E plus g minus.

Now what do I do is I rearrange. I bring this over to this side, this over to this side. And I can still use linearity because this is an integral of non-negative functions. I get that the integral of E over f plus g plus minus the integral of E over f plus g minus is equal to-- so let's just do this slowly. This is equal to this integral and this integral.

So even though we didn't know that the positive part and-- equals the sum of the positive parts, so even though we don't know that the positive part of the sum is the sum of the positive parts and the same thing for the negative part, we still get the integrals equal. And this thing right here is just equal to, by definition, the integral of f plus g. And this thing over here-- again, if we use linearity for the integrals of non-negative functions, this is equal to f plus plus the integral over E of g plus, minus-- and again, just expanding this and then carrying through the minus, minus F minus.

So you see here, all we're using is linearity for non-negative for a sum of two non-negative measurable functions. And this is just, again, by the definition of the integral the sum of the integrals.

And so that was 2. What's the proof of 3? 3 from 2 and the fact that if I take f times the indicator function of A union B, this is equal to f times the indicator function of A plus f times the indicator function of B when A and B are two disjoint sets. So using what we know about for the integral of non-negative measurable functions, it follows that the integral of f over A union B is equal to the integral of this quantity here over E, which is equal to the sum of the integral of this sum here, which is equal to the sum of the integrals.

And then going back, that's equal to the integral of f over A plus the integral of f over B. So again, it follows from linearity, this fact, and although I didn't write it down, the fact that even now for integrable functions, but that the integral of a subset is equal to the integral of the function times the indicator. The reason that's true is because it's true for non-negative measurable functions and simply from the definition of the Lesbesgue integral.

All right, so some more properties of the integral. Suppose f and g from a measurable subset to R are integrable. Then the following holds. The integral-- I mean, the absolute value of the integral is less than or equal to the integral of the absolute value. If f equals g almost everywhere, then-- so let me-- so let's just-- I have no idea what I was going to say. I had to cut it down a little differently. So let's restart.

Suppose f and g are two functions measurable. Need to write measurable. OK, back from the top, if f is integrable, then the absolute value of the integral of f is less than or equal to the integral of the absolute value. If g is integrable, and f equals g almost everywhere, then f is integrable. And the integral of f over E equals the integral of g, again, back to that philosophy from last lecture where I said if your conclusions are in terms of integrals, then usually the hypotheses can be stated only in terms of almost everywhere information.

3, if f and g our integrable, and f is less than or equal to g almost everywhere, then the integral of f over E is less than or equal to the integral of g. So one follows simply from the definition and the relationship between the absolute value of f, f, and the positive and negative part. So we have the absolute value of the integral of E of f over E of f is equal to, by definition, the integral of the positive part of E minus the negative part of E.

Now these are two non-negative numbers. So by the triangle inequality, the absolute value of their difference is less than or equal to the sum of the absolute values, which are equal to themselves because they're non-negative numbers. And now the integral of-- or the sum of integrals is equal to the integral of the sum.

And f plus the positive part plus the negative part of f is equal to the absolute value of f. So the triangle inequality-- or not-- I shouldn't say triangle inequality. But I mean, if you think of the integral as the sum, then this is a version of the triangle inequality. But the triangle inequality for Lesbesgue integrals holds.

So 2, so we have f is less than or equal to-- so first off, absolute value of f is equal to the absolute value of g almost everywhere, which implies what we know from the integral of non-negative measurable functions, that if I have two non-negative measurable functions who equal each other almost everywhere, then their integrals agree. And that's fine. Thus, f is integrable.

Now why does the integral of f over E equal the integral of g over E? So since f equals g almost everywhere, f minus g in absolute value is equal to 0 almost everywhere. And therefore, if I look at the absolute value of the difference of these two integrals, this is by linearity equal to the integral of the difference, which by the triangle inequality for Lesbesgue integrals is less than or equal to the integral of the absolute value.

And by what we know for the integral of non-negative measurable functions, if I have a function which is 0 almost everywhere, then its integral is 0. And therefore, the absolute value of-- or the integral of f over E must equal the integral of g over E. So that proves 2.

And 3 is-- again, so we're just using the stuff that we know from the integral of non-negative measurable functions. Find a function h of x to be g of x minus f of x and 0. Now when is it this? This is when if g of x is bigger than or equal to f of x, and 0 otherwise.

So this is a non-negative measurable function. It's equal to g of x minus f of x when g of x is bigger than or equal to f of x, and 0 otherwise. And h is a non-negative measurable function. h equals-- since g is bigger than or equal to f almost everywhere, h is equal to g minus f almost everywhere because this condition is satisfied almost everywhere.

And therefore, I get that 0, which is less than or equal to the integral over E of the positive part of h-- because this is just a non-negative function, we know what its Lesbesgue integral is defined to be. It's just given by the previous lecture. So it has to be a non-negative number, which because h is non-negative, this is equal to h. h plus is equal to h.

But what we proved a minute ago, since these two functions, integrable function, so I should say why. So why am I also asserting that h is an integrable function? Well, the integral of h is equal to the integral of g minus f, which, if I want to put absolute values on things, the integral of the absolute value of h is equal to the difference of two integral functions, which is integrable almost Everywhere and by what we did for part 2, then that tells me h is integrable.

So the integral of H, so by 2, must be equal to the integral of g minus f, which by the linearity that we proved in the theorem before is equal to the integral of g minus the integral of f. And you remember, we started all the way at the end of this or at the beginning of this with 0 is less than or equal to. So the integral of g will be bigger than or equal to the integral of f.

So now the final most useful convergence theorem one encounters in Lesbesgue integration or in integration theory is Lesbesgue's dominated convergence theorem. Or I'll just call it the dominated convergence theorem. Let me make a small comment. Well, let me pause and make a small comment right here.

What functions are Lesbesgue integrable first off? What are some examples of functions that are Lesbesgue integrable? Did I-- let's see. Did I skip that? Oh, I had it at one point. So what sets have finite measure?

We know that the set-- the measure of intervals is the length of the intervals, right? So any sets that are contained in a large interval are-- any measurable subsets that are contained in a large interval have finite measures. So compact sets, which we know are measurable, because they're Borel sets, are-- have finite measure. So why am I saying that?

If I have simple functions which are nonzero only on sets that have finite measure, then that will be an integrable function, because go back to the definition of how one integrates a simple function, right? It's the sum of the coefficients times the measure of the sets, where the-- it takes that coefficient. So if I'm only nonzero on a set of-- with the convention that 0 times infinity equals 0 so that it's the measure of 0 times the measure of where the function is 0 is equal to 0, then those simple functions that are nonzero only on sets of finite measure will be integral functions.

Now what about continuous functions? So in fact, let me-- I'm not going to prove this because we're going to prove something much stronger in a minute. So what about continuous functions on a closed and bounded interval, say a-- so a continuous function on AB? Let's just talk our way through why those functions are integrable on those sets.

So a continuous function on an interval AB-- if a function is continuous, then its absolute value is continuous on AB. And therefore, the absolute value must be bounded by some constant. A continuous function attains a minimum and maximum on a closed and bounded interval. So the absolute value of a continuous function on a closed and bounded interval is bounded by some constant.

And therefore, by monotonicity of Lesbesgue integrals, the integral of the absolute value of f over the interval AB will be less than or equal to some constant times the interval over the-- or times-- will be less than or equal to the integral of the constant over AB. And the integral of a constant is just equal to the constant times a measure of the set. Just by how we define the integral of simple functions, a constant is just the simplest of simple functions, which would be the constant bounding above the absolute value times the measure of the closed and bounded interval AB, which is B minus a

So the Lesbesgue integral of a continuous function-- so what was all that for? To say that a continuous function on a closed and bounded interval is a Lesbesgue integrable. Now we're about to show something right after I prove the state and prove the dominated convergence theorem, which is much stronger than that, is that in fact, the Lesbesgue integral of a continuous function on the closed and bounded interval equals its Riemann integral. So in fact, for continuous functions, you already know how to compute Lesbesgue integrals. They're equal to the Riemann integrals.

So back to the dominated convergence theorem. Let g be an integrable function over E. f in a sequence of measurable functions such that-- two things. One, for all n, the absolute value of fn is less than or equal to g. So g is non-negative. I guess I should have put here non-negative. And almost everywhere, there exists an f from E to R such that fn converges to f pointwise is almost everywhere, so meaning fn of x converges to f of x for almost every x in E.

So these, the sequence of measurable functions, converge to-- pointwise to this function almost everywhere. And they're all dominated by an integrable function. Then the conclusion is that the limit of the integrals equals the integral of the limits or the integral of the limit.

So this is a very useful and powerful theorem in integration. It's way stronger than you can-- than anything one can really say in Riemann integration, as far as convergence theorems go. Riemann integration requires always in some form, some form of uniform convergence here. All we have is pointwise convergence almost everywhere, right?

And the second requirement-- so remember, the monotone convergence theorem also required the functions to be increasing. So here, now we're dealing not just with non-negative functions, just arbitrary measurable functions. Then I should say real valued measurable functions because our measurable functions could also be extended real value. But I wanted this to be real value.

So all we need on top of pointwise convergence almost everywhere is just for these to be bounded above by some integrable function, some fixed integrable function. Then we conclude that the limit of the integrals is equal to the integral of the limit, which is an extremely powerful and useful theorem in analysis. So we'll prove it using Fatou's lemma.

But first, again, the conclusions are about integrals. And I'm making almost everywhere statements in the assumptions, which I said you could always basically do. Let me briefly reduce ourselves to the case that these two things hold almost everywhere. I mean everywhere. Since for all n, fn is less than or equal to g almost everywhere, this implies that for all n, fn is integrable.

So moreover, fn converges to f almost everywhere implies a couple of things, that-- remember, the pointwise even almost everywhere convergence of measurable functions is a measurable function. So f is measurable. And so f is less than or equal to g almost everywhere, which implies that f is integrable.

So since changing fn for each n-- or I should say-- let me say this. Since changing f on a-- what am I saying? Since changing f and fn for all n on a set of measure 0 does not affect the integrals.

In the end, our conclusion should be about the limit of the integral of the sequence equals the integral of f. And since if I change fn and f for each n on a set of measure 0, doesn't change the integrals, right? We can assume that those two assumptions-- we can assume that these two-- we can assume that these two assumptions hold everywhere. And these hold everywhere on E.

So first off, I want to note-- so now let's actually get to the proof. All right, maybe you need to take a second to think about what I said here. But the point is that because the conclusions are in terms of integrals, and although I stated it for almost everywhere, I can fiddle with fn and f for each n on a fixed set of measure 0 without affecting the integrals.

So if I want to prove this, I can change them on a set of measure 0 without affecting the integrals. And on the set of measures 0, I make it so that all of the fns equal f. And so then I have this convergence everywhere. And I change f to be 0 on that set as well so that I will have this everywhere as well.

But so if you need a second, if you'd like, just imagine that I erased that "almost everywhere" so that the statement of the theorem has these things holding everywhere. And then think a little bit about why I don't need it to hold everywhere, just almost everywhere.

So we assume for all n, fn is less than or equal to g. There exists an f so that I have that. So note, for all n, the integral of f sub n is less than or equal to the integral of the absolute value, which is less than or equal to the integral of g, which implies that the sequence formed by the integrals of these guys-- so this is a sequence of real numbers-- is a bounded sequence of real numbers. So it has a limsup and a liminf, right?

Now remember from your first analysis course, whatever it was, that the limit of a bounded sequence is equal to L if and only if the liminf and limsup equal each other, and they equal L. So what we're going to do is we're going to show that the liminf and the limsup of this sequence of numbers are equal. And they equal the integral of f.

And to do that, we're going to use Fatou's lemma. So since g plus or minus f sub n is less-- or is bigger than or equal to 0 because the absolute value of f sub n is always less than or equal to g. So g plus or minus f sub n is always bigger than or equal to 0. I can now apply Fatou's lemma, which tells me that for a sequence of non-negative measurable functions, the integral of the liminf is less than or equal to the liminf of the integrals.

So again, so the fns are converging to f. So f of x-- fn of x for all-- so fix x in E, fn of x converges to f of x. Therefore, liminf of fn of x equals f of x. So the liminf of g minus f equals fn. Sorry, equals g minus f because fns converge to f, again, for each pointwise. Let's do a minus here.

And by Fatou's lemma, this is less than or equal to liminf as n goes to infinity of the integral of E-- over E of g minus f sub n. Now this is equal to the integral. Using linearity, this is equal to the integral of g minus the integral of f sub n.

When I take the liminf of that and carry it through, I get the liminf when it hits a minus turns into a limsup. So whenever I have a bounded sequence of numbers, the liminf minus that sequence of numbers equals minus the limsup of that sequence of numbers. So here I'm using liminf of minus An, n being-- equals minus limsup of An.

And similarly, I get that the integral of g plus f is less than or equal to just by-- now if I choose lemma again, but now I don't have to switch minuses, this is less than or equal to the integral of g plus liminf as n goes to infinity of the integral of fn of E. So I have two inequalities. I have that. And I should say it's less than or equal to, so maybe include a little bit of that. And then I also have this inequality here.

Now all of these quantities that I've written down, these are all finite numbers. This is one reason why. So this is the limsup of this bounded sequence. That's a number. This is a number. That's a number. So I can subtract and move them to either side of this inequality. So there's no funny business going on with subtracting infinities. This is all on the level.

So moving the limsup over and subtracting over the integral of g minus f, I get that the limsup as n goes to infinity of f sub n is less than or equal to the integral of g minus f. And now by linearity, that's equal to the integra of f. And this is also equal to, by linearity-- let's see-- minus the integral of g. And by the second yellow box here, this is less than or equal to the integral of g plus the liminf of the fn, so liminf of the integrals of the fns.

So what do I have? I have the limsup is less than or equal to the integral of f, is less than or equal to the liminf. The liminf always sits below the limsup. So therefore, those three numbers have to equal each other. So this box always sits below this box. So all three numbers must equal each other then-- equals the integral of f equals liminf as n goes to infinity of fn. And I bet in analysis, you thought limsups and liminfs would never be useful, but they are.

So that is the proof of the dominated convergence theorem. Now let's use some of this muscle we've been building up. So suppose a is less than b and f is a continuous function on a, b. Then the Lesbesgue integral over a, b of f is equal to the Riemann integral of f. This is the Riemann integral. So in the course of this proof, we'll also see why f is, in fact, integrable.

So, proof. So we first show f is Lesbesgue integrable. So this implies that the absolute value of f is also a continuous function. And every continuous function on a closed and bounded interval is bounded. So there exists a constant b such that f is-- the absolute value of f is less than or equal to b on this closed and bounded interval.

Then the integral of the absolute value of f, the Lesbesgue integral of f of the absolute value of f over a, b, this is less than or equal to a times b, the integral over a, b of capital P. And this is the simplest of simple functions. The Lesbesgue integral of this is just b times the measure of b of a, b, equals b times b minus a, which is finite. So continuous functions are Lesbesgue integrable on a closed and bounded interval. Thus, f is Lesbesgue integrable.

Now the positive part of f is a continuous function. And the negative part of f is a continuous function. So in fact, you can write these down a little bit differently than I wrote them down before. f plus is equal to-- let's see. So these are the positive and negative parts written slightly differently. If f is continuous, both of these functions are continuous non-negative functions.

And the Riemann integral of f plus minus f minus, which is f, the integral of f, is equal to the integral of f plus minus the integral of f minus, which is exactly how also Lesbesgue integrals are defined in terms of the integral, Lesbesgue integral of f is equal to the Lesbesgue integral of f plus minus Lesbesgue integral of f minus. If I consider simply these two cases separately, be considering these separately and showing the integral over a, b, f plus or minus equals the corresponding Riemann integral. And using linearity, I may assume that f is non-negative.

So what's the point here? I'm trying to show this for general continuous functions. But by splitting it into its positive and negative parts, it suffices to prove this equals that for the positive and negative parts, both of which are continuous functions and non-negative. So I only need to prove what I want for the case that f is non-negative. That's the point.

All right, so we now have a non-negative continuous function on a, b. And we want to show the Lesbesgue integral is equal to the Riemann integral of that continuous function. So let xn be-- let's see. So this should equal a. This should equal b-- be a sequence of partitions of a, b such that the norm of-- so this is just notation from back in real analysis.

You shouldn't take this as an actual norm. Well, I guess it is a norm in a certain sense. But this is just a subset of a, b that partitions a, b such that this quantity here, which I denote using the norm-- but don't confuse us with norms that we discussed before, which is defined to be the max of-- and m could change with n. Goes to 0.

So why am I taking a sequence of partitions of a, b? Because this is how you compute the Riemann integral in terms of Riemann sums. And I'm going to show that the sequence of Riemann sums converging to the Riemann integral actually converges to the Lesbesgue integral as well, but along a certain sequence of Riemann sums.

So let xi j, n in one of the subpartitions. So f is a non-negative continuous function on this interval. So it has a minimum that it achieves at some point. Equals f of xi j, n. So on the subinterval, f of x is always bigger than or equal to f of xi n, j. So this is how I'm defining the xi n.

Then by the theory of Riemann integration, if I look at limit as n goes to infinity of the associated Riemann sums, this limit exists. And you get the Riemann integral of f for a continuous function. This should have been covered in your introductory analysis class that-- if you like, this is the lower Riemann integral or lower Riemann sum. And that as long as you're going along a partition so that this quantity here is going to 0, then the associated Riemann sums converge to the Riemann integral.

All right, now each of these is a finite set. Let n be the union of these sets. Then this is a countable union of finite sets. So it's countable. A countable union of countable sets is countable. In particular, this means the measure of this set is 0.

So if I-- the set of all partition points as I range over all of the partitions converging to 0-- so I just took any sequence of partitions with this quantity here going to 0. If I take the union of all these partitions, I get a countable set. That set has measure 0 because it's countable.

Why am I making this point that it has measure 0? Well, because off of this set, magic happens. And what we've learned is that magic happening off of a set of measure 0 means magic happens for integrals. So let me-- I'd like one more important piece of information that we have from the theory of Riemann integration that the Riemann sums converge to a Riemann integral.

Let fn be the following simple function. This is sum from j equals 1 to mn of f of c in j of the indicator-- times the indicator function of xjn j minus 1 xjn. And then could put plus 0 times the indicator function of xjn. I mean, this part doesn't really matter. I'm just saying.

So this is a simple function for each n. Should say a non-negative simple function. So what's happening now? And why did I choose the xi n? So let me draw the picture that goes with this.

I have my function f on a, b. And what I'm doing is I'm cutting up the domain to get the approximate Riemann integral. So that should connect. And I'm choosing the heights to be the minimum of f on each of these intervals.

So this is a f of xi 1, if you like, f of xi 2. And at least for this picture, this is xi 1. And what I know is as I'm making this partition finer and finer, these approximate areas here are converging to the full Riemann integral of f.

Another way to think about that is that what these are just the Lesbesgue integral of certain simple functions, where the simple function is 1 on this and has height f of xi in or xi 1. This quantity here, the integral-- the area underneath this-- is equal to the Lesbesgue integral of f of xi 2 times the indicator function of this interval, and so on. So I can view these pieces here that are entering in the Riemann sum as Lesbesgue integrals of certain simple functions. Or I can view this entire quantity as the Lesbesgue integral of a simple function for reach n.

Now the goal here is what we'll do is we'll show that the simple functions whose-- I mean, in fact, we can do just do this now. Note, for all n, if I look at the Lesbesgue integral of fn over a, b, this is equal to sum from j equals 1 to m sub n. This is a set of measure 0, so it doesn't contribute. f of xi jn-- these are all non-negative numbers because f is non-negative-- times the measure of xj minus 1n, xjn times xjn.

Now remember, we built up Lesbesgue integration so that the measure of an interval is the length of the interval. So this is equal to xjn minus xj minus 1n. So the integral, the Lesbesgue integral, of each of these functions, each of these simple functions, equals that Riemann sum appearing in this limit, right?

And now the goal is to show that the fns converge to f almost everywhere and are bounded above by an integral function at least almost everywhere. Then I can apply the dominated convergence theorem to conclude that the limit n m goes to infinity of this thing equals the Lesbesgue integral of f. But this, the limit as n goes to infinity of this thing, is equal to the limit as n goes to infinity of this quantity, which is equal to the Riemann integral. And that's the game plan.

And I mean, you can already guess from here. What's the function that sits above all of the fns? It's going to be at least away from these-- at least away from possibly the endpoints. It's going to be f. Then for all xn a, b, take away n. A couple of things I have. Well, first thing is less than or equal to f of x.

Now I'm going to show that on a, b, take away n, which is a set of measure 0, that fn of x converges to f of x. So now I claim-- then-- i.e., fn converges to f almost everywhere. And it's bounded above by a Lesbesgue integral function almost everywhere. So then we can apply the dominated convergence theorem to get what we want. So let's prove this.

So let xb and ab take away the set of all partition points. So we want to show this. Let's just go back to a basic epsilon. In argument, let epsilon be positive. Since f is continuous at x, there exists a delta positive so that if x minus y is less than delta, then f of x minus f of y is less than epsilon.

Now we know that the partitions are getting finer and finer, right? So x is in a, b take away n, right? So since the norms of these partitions, which remember, is the max-- I forgot that n when I was writing it here. And this-- should be n. Since this goes to 0, there exist capital M so that all n bigger than or equal to capital M, this quantity here, the length of-- the longest length of the subintervals-- is less than delta.

So now let me draw you a picture. So x is in a, b, take away the partitions, all the possible partition points, all the xj's, n's, right? So let-- I claim now that fn of x minus f of x is less than epsilon. Now how do I evaluate for x in a, b take away n fn of x? And fn of x is equal to-- times the indicator function of xj minus 1n, xjn of x.

And I don't have to worry about the endpoint, because remember, x is in a, b, take away all of the partition points. xjn is always a partition point. xjn is always b, so I'm always taking away b and a. So this must equal-- this x must lie in one of these intervals, but not be one of the partition points. So this must equal-- this is bad notation. But let's say fxe, OK, k-- for the unique k, such that x is in xk minus 1n xk.

Then since xi n, k is in this interval, and the max over the length of the small intervals, which is the differences here, is less than delta-- so in particular for this one. So I have-- here's a picture. Here's xk minus 1n, xnk. Xi nk is somewhere in there. x is somewhere in there.

And since this quantity here is less than delta, this implies that x minus xi in k must be less than delta. And therefore, f of x minus f xi f sub n of x, which is equal to f of x minus f of xi kn. Now these are within delta distance to each other.

And therefore, f of x and f of that number must be within epsilon of each other simply by how delta was chosen. So we've shown that for all n bigger than or equal to capital M, f of x minus fn of x is less than epsilon. Thus, limit as n goes to infinity of fn of x equals f of x, all x in a, b, take away the partition points.

So we have a couple of things. So I said this, but now I'm just going to write down what. We have these two things. We have that almost everywhere, fn's are converging to f. Almost everywhere, fn is less than or equal to f. f is a fixed continuous function that's integrable. It's non-negative.

So by the dominated convergence theorem, the integral, Lesbesgue integral, of this function, continuous function f of x, is equal to the limit as n goes to infinity of the Lesbesgue integrals of these special simple functions, which, as we computed right here, is equal to the limit as n goes to infinity of k equals 1 to m sub n, f of xi in j minus xj and minus xj minus 1n. And by how this-- what we know about Riemann integrals, this converges to the Riemann integral. And thus the Lesbesgue integral equals the Riemann integral for a continuous function.

So we have discussed the Riemann integral of real-valued measurable functions or real-valued integrable functions. Now quite often, we want to have complex-valued functions defined on measurable subsets of real numbers. What do we do then? Well, everything that we've done basically applies as long as it makes sense. So let me state that here.

So all of the previous theorems that we've proven, as long as they make sense, you can use what we've done for real-valued integrable functions to what we call complex-valued integrable function imply the corresponding statements for complex-valued integrable functions. And what are these? So f from a measurable subset of E to now the complex numbers.

So let me say E to-- and now let me just define what I mean by complex integrable functions, meaning-- so f from a measurable subset of E to C is Lesbesgue integrable if the same condition holds, that the integral of the absolute value of f is-- or now this is a modulus of a complex number for each x-- is integrable. So this is a non-negative measurable function now which is defined on E. So this quantity here makes sense. So we say a complex-valued function is Lesbesgue integrable if this quantity is finite.

And now the definition of the Lesbesgue integral of a complex-valued integrable function is defined to be simply the integral of the real part of f, which is, again, a real-valued integrable function, plus i times the integral of the imaginary part of f. So again, we just-- to define complex integrable functions or complex-valued integrable functions, we just take the same definition. And then we define the integral to be the integral of the real part plus i times the integral of the imaginary part. And the fact that this is finite implies that these two real-valued measurable functions are, in fact, integrable.

And all of the theorems that we stated before, as long as they make sense, carry over. For example, linearity of the integral with respect to now multiplication by complex numbers carries over. The integral of the sum is the sum of the integrals. That still carries over just by using this definition, along with the fact that we know that the integral of two-- the sum of two real-valued functions is integrable. And the Lesbesgue dominated convergence theorem then carries-- can be generalized to complex-valued integrable functions.

Maybe what's not so clear is the-- or let me just at least give you the flavor of how you can use what you know about real-valued integrable functions to get corresponding statements for complex-valued integral functions. So let's do the triangle inequality for integrals, right? So if I have a complex integrable or complex-valued integral-- complex-valued integrable function-- then the integral of E of f-- so this is now a complex number. Taking the modulus of that is less than or equal to that.

So what's the proof? So this is clear. So if the integral of E equals 0. So that's just clear automatically. So let's assume it's not equal to 0. And let alpha be the following complex number, integral of E over-- so the integral of f over E is a complex number. I take its complex conjugate. I divide by the modulus of that.

Then alpha and modulus equals 1. And the absolute value of the integral of f-- this is equal to alpha times the integral of f over E because the complex conjugate times this gives me the modulus squared divided by the modulus. I get back the modulus, right? And linearity of the Riemann-- or not Riemann-- of the Lesbesguen rule for complex-valued integrable functions still holds.

So I can pull that alpha inside. And now this thing is equal to this. So it's equal to a real number. So it's equal to its real part. And now the real part of the integral is equal to the integral of the real part just by the definition. So this is equal to the real part of alpha times f.

Now this is a real-valued function, integrable function. And we know from what we proved for real-valued integrable functions, that's less than or equal to the real part of alpha times f over E. Now the absolute value of the real part of a complex number is less than or equal to the modulus of that complex number. And alpha has modulus equal to 1. So this is equal to the integral of the absolute value.

So at this point right here, we used what we knew about real-valued integrable functions. And using that, we then got the corresponding statement for complex-valued integrable functions. And using the theorems that we proved for real-valued integrable functions, we get corresponding statements for complex-valued integrable functions as long as the statements make sense.

We don't say two complex-valued functions, one is less than or equal to the other, because we don't have an order on complex numbers. So that statement doesn't make sense. But statements about, for example, complex-valued continuous functions, the Lesbesgue integral equaling the Riemann integral, that follows just immediately because the Riemann integral of a complex number-- a complex-valued continuous function-- is just defined to be the same thing. And since we know about-- know that this equals the Riemann integral of the real part of f, and this equals the Riemann integral of the imaginary part of f, which is by definition equal to the Riemann integral of f, then we immediately get the previous theorem generalized to complex-valued functions.

All right, so that's the theory of Riemann integration. I mean-- Riemann integration? Lesbesgue integration. Next time, we will finish our discussion of-- measure an integration by introducing the big Lp spaces, which are spaces of measurable functions which have a finite-- which raised to a certain power, have a finite Lesbesguen rule, and show that those are Banach spaces that contain the continuous functions, of course, and in certain cases are-- the space of continuous functions are dense in these spaces.