[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIGUEZ: So let's get started. Last time, at the end of the last lecture, we introduced outer measure. So we first discussed what we wanted to measure, the properties of kind of a notion measure of subsets of to satisfy.

We wanted it to, first off, be defined for all subsets. We then wanted the measure of an interval to be the length of the interval. We also wanted the measure of a union of disjoint subsets to be the sum of the measures, and said we wanted it to be translation invariant, meaning if I take a set and I shift it by a fixed amount, then the measure of the shifted set is the same as the measure of the original set.

But as I said last time, that's impossible. You can't. There does not exist such a thing defined on every subset of real numbers.

So what we're going to do is define-- or what we're doing right now is defining outer measure, which will satisfy almost all of the properties we wanted, and is defined for every subset of R. And then we'll restrict this outer measure to a certain class of subsets of R that are well behaved with respect to measure. And then we'll get a function, now defined on a collection of subsets of R, what we'll call measurable sets.

And this set function, which we call measure, Lebesgue measure, will satisfy the three properties, the main three properties we wanted-- that the measure of an interval is the length of the interval, the measure of a countable disjoint union of sets is the sum of the measures, and it's translation invariant. Now at the end of last time-- so we defined outer measure and we also proved this theorem that if we have a countable collection of subsets of R, then the measure of the union is less than or equal to the sum of the measures.

Now, what we would like in the end, like I said, is to be able to have equality whenever the subsets are disjoint. Now, this doesn't hold for outer measure, but as we'll see, once we restrict outer measure to a certain class of subsets, we will get the property that we want. Now outer measure almost satisfies one of those properties we want to have of a measure.

What we're going to verify now is that outer measure does satisfy one of the properties we wanted of a measure, namely that the measure of a interval ought to be its length. So that's the following-- if I is an interval, then the measure, the outer measure of I, is equal to the length of I. So here, the length of I-- remember, if it's an interval of the form a, b, with the endpoints included or not, the length of the interval is b minus a. If it's an infinite interval, then the length is infinite.

So the most involved part, or at least the crux of proving this, is doing the simplest kind of interval, which is a closed and bounded interval, a, b. So let's do that case first. Suppose I is equal to a, b.

And also, something that I didn't write down here is we have, immediately from the definition of outer measure-- so let me just pause that real quick-- that if A is a subset of B, then the outer measure of A is less than or equal to the outer measure of B, simply because if I take any open collection of open intervals covering B, then that will be a collection of open intervals covering A. And since infimums of bigger sets-- let's see, which direction is that-- since infimums of bigger sets have to decrease, that gives me the inequality that I want.

So going back to the proof of this theorem, let's suppose I equals a, b. What we're going to show is that the outer measure is less than or equal to the length of I, and then the converse-- that the length of I is less than or equal to the outer measure of I. So the simpler one is showing the outer measure of I is less than or equal to the length of I.

So then I is contained in the open interval, the single open interval, I minus epsilon, b plus epsilon for all epsilon positive. And therefore, since the outer measure of I is the infimum of the sum of lengths of intervals covering I, this implies that the outer measure of I is less than or equal to the length of this interval, which is equal to b minus a plus 2 epsilon for all epsilon positive. And therefore, if I have this is less than or equal to b minus a plus to epsilon for all epsilon positive, I can then send epsilon to 0 to conclude that the outer measure of the interval is less than or equal to b minus a, the length of the interval. So that's the simple part.

So now, we're going to show that b minus a is less than or equal to the outer measure of the interval. So also, what this shows is that the outer measure of this interval is finite. It's a finite number.

So to show this, what we have to show is that if I take any collection of open intervals covering I, then the sum of the lengths of those intervals is bounded below by b minus a. And if I don't write n equals 1 to infinity, it should be understood that n is always going from 1 to infinity. Let I n be a collection of open intervals such that this interval I, a, b, is contained in the union. We now want to show that b minus a is less than or equal to the sum of the measures of the I n's.

Now, the closed and bounded interval, a, b-- this is a compact set. So you look back to 100B, which is covered by a collection of open intervals-- in other words, open sets. The definition of compactness-- so since this closed and bounded interval is compact, this is due to a special theorem by Heine and Borel-- we can find a finite covering of this interval by finitely many of these intervals.

So I can choose finitely many of these intervals to cover a, b. There exists a finite collection, which I will now denote, let's say, J k, k equals 1 to n, so contained inside this collection of open intervals-- so these are just finitely many open intervals from this collection-- such that a, b is contained in k equals 1 to n J k.

So now, I have these open intervals. And now what I'm going to argue is that I can do this-- is that here's the plan. Here's a. There's b. What I'm going to argue now to you is that rearranging how I'm indexing these open intervals, that I can cover a, b in the following way-- so this is the first.

So I'm going to argue that we can cover this interval like this, so that I can choose the first interval to cover some of a, b, and then maybe it covers all, in which case I would stop my construction. If it doesn't cover all of a, b, then I can cover it with J2. So now, I have a2, b2.

And if that doesn't cover all of a, b, I will still have to choose some more intervals to cover it. And eventually, I'll get to-- at least in this picture, since I can't draw n intervals-- I will be able to cover a, b in such a way that these are kind of linked together, so that these intervals are linked. And then I cover all of a, b.

Now, why is that great? Well, then, because the sum of the lengths of these intervals is going to be bigger than or equal to-- the sum of the lengths of these intervals is going to be what? b3 minus a1, which is bigger than or equal to b minus a. And that would give me the lower bound that I desire.

So now, let me argue that we can do this-- that this picture is correct. Since a is in this-- it's in a, b, so it has to be in this finite union of open intervals-- there exists k1 such that a is in J of k1. So one of these intervals, a has to be in it. And by rearranging the intervals, I can assume that k1 equals 1, i.e., a is in J1, which is a1, b1.

Now, it's possible that this whole interval covers a, b, in which case I would stop the construction. Otherwise, I continue. If b1 is less than or equal to b, then b1 is certainly in this interval, which implies there exists a J k sub 2-- let me write it this way-- there exists k sub 2 such that b1 is in j sub k sub 2. And again, by rearranging, we can assume that k sub 2 equals 2.

So by re-arranging the remaining intervals, I can assume k2 is 2. So if in the case, like I've drawn in the picture, b1 is not bigger than b, then I can find another interval J2 that contains b1. So b1 is in J2, which is a2, b2.

And I will just continue this until the endpoint of one of these intervals passes b. The first instance when one of these intervals passes b, I stop this process. And it has to stop because all of these intervals do cover b, so it has to occur at some point. And there's only finitely many intervals. And I'm going to write if b2 is less than or equal to b, but I'm going to put dot, dot, dot there.

So what have we done by this argument? Thus, we conclude that-- maybe that's a little bit bad notation because I have n there. So let's make this a capital N. Sorry about that. Make that a capital N, since that little n appears also to index the I sub n's.

So thus, we conclude that there exists a k with 1 less than or equal to k less than or equal to n such that three things hold. For all k equals 1 to k minus 1, we have that b k is less than or equal to b and a k plus 1 is less than b k is less than b k plus 1. So think of this as a condition where we have not yet covered the entire interval a, b.

For k equals 1, this was basically the argument I just gave here. And two, that this has to stop at some point, so that b is less than b capital K. So what I drew here-- so the picture is for the K is K equals 3.

So we have this K, and I drew the picture over there for b3. Now I'm going to show that the sum of the links of the intervals-- and this is just a finite collection from the bigger collection of intervals-- is now bounded by b minus a, which is kind of clear from the picture that I drew. But we still have to write stuff down. This is not Topology.

Then sum of length of I sub n's-- this is certainly less than or equal to the sum over this finite subcollection of intervals, lengths of J k. And this is less than or equal to sum k, K-- now here, this capital K is coming from over there-- length of J k. And now what is this? This is equal to b K minus a K plus b K minus 1 minus a K minus 1 plus, and then all the way down to b1 minus a1.

But here's the trick-- remember, each of the b K minus 1's, they lie ahead of a K. So the index here is shifted just by a little bit. But this says that the previous b is in front of the next a.

So I can collect terms and write this as b K plus b K minus 1 minus a K plus b K minus 2 minus a K minus 1. So I just borrowed a b coming after this term, so on, and so on, until I get b1 minus a2. And then I'm stuck with a a1.

And by the second condition over here, everything in parentheses is non-negative. So this is bigger than or equal to b K minus a1. And what do we know? We know b K is bigger than b, and we know a1 is less than a because a is in a1, b1. So this length is bigger than b minus a.

And since we've shown that the sum of lengths of intervals covering I is bigger than or equal to b minus a no matter what the collection of intervals is, then the infimum has to be bigger than or equal to b minus a. And we conclude that the outer measure of this interval a, b, is bigger than or equal to b minus a. And therefore, I have both sides of the inequality I want, and therefore, I have equality.

Now, this was for a closed and bounded interval. But this essentially gives us the result for any interval. So if I use any finite interval of the form a, b, a, b, not including a, open, and then for all epsilon positive, at least sufficiently small, what do I have?

I have that a plus epsilon, b minus epsilon is contained in I, which is contained in a minus epsilon b plus epsilon. I is one of these intervals. So if I just make it a little fatter and take that closed interval containing it, I get this side. And then if I just kind of shrink the interval a little bit and take the closed interval, this guy will be contained inside of it.

And therefore, the measure of this interval has to be less than or equal to the outer measure of this interval, which is less than or equal to the outer measure of this interval. And I get that-- now, the measure of this interval-- I'll just write this.

And this over here is b minus a minus 2 epsilon is less than or equal to the outer measure of the interval, which is less than or equal to the outer measure of b minus a plus 2 epsilon. And this holds for all epsilon positive. Recall that I was starting off for all epsilon positive, at least sufficiently small depending on b and a, smaller than the difference of b and a.

And therefore, if I send epsilon to 0, I get b minus a is less than or equal to the outer measure of this interval, this finite interval is less than or equal to-- what why do I have this here? Is less than or equal to b minus a. And therefore, the outer measure of any one of these finite intervals is the length of the interval.

And then I'm going to leave it to you as an exercise, which is not difficult. If I is equal to R negative infinity to a, a infinity, or include-- then the outer measure of this interval is infinity. In other words, I cannot ever cover these intervals by a countable collection of intervals whose sum of lengths is a finite number. Again, this is not hard.

So we get kind of a nice, little fact from the theorem we proved last time, which was that if I have a collection of subsets of R, then the measure of the union is less than or equal to the sum of the measures, and this theorem, which says that the measure of an interval is the length of the interval, which is the following-- for every subset of R and epsilon positive, there exists an open set O such that A is contained in O and the measure of A, which is less than or equal to the measure of O simply because A is contained in O, is less than or equal to the outer measure of A plus epsilon.

So somehow, the outer measure of subsets can be approximately measured by the outer measure of open sets. So another way to say this is, at least with respect to outer measure, every set can be approximated by an open set, an outer measure. And what's the proof of this?

So this is clear if the outer measure is infinite. Then I just take O to be the entire real number line. So suppose it is finite. Let I n be a collection of open intervals such that they cover A.

And remember, the outer measure is the infimum. So I can get arbitrarily close to that infimum by summing lengths of certain collections of open intervals. And sum of the links is less than or equal to, if you like, the outer measure of A plus epsilon. So I should have said at the beginning, let A be a subset of R and epsilon be positive, but you understand.

And so what I do is I just take O to be this union of open intervals. So O is a union of open intervals. Each of these open intervals is an open set.

And you should know that the union of open sets, any collection of open sets, is, again, an open set. So this is open. A is contained in O because A is contained in the union.

And the outer measure of O, which is equal to the outer measure of this union of open intervals is, by the theorem we proved last time, which I stated with the definition, or when I recall the definition is less than or equal to the sum of the outer measures of the intervals, which we just proved is the length of these intervals. And how we chose these intervals, remember, is so that we have this condition. And this is less than or equal to-- so with respect to outer measure, every set can be approximated by the approximation with respect to outer measure by an open set, by a suitable open set.

So we've defined outer measure. We've proven some properties of it. Now, we're in a position to at least define measurable sets.

And I should say these are Lebesgue measurable sets. So a subset of real numbers is Lebesgue measurable-- so this is a new piece of terminology-- if for all subsets of R, if I look at the outer measure of A, this is equal to the outer measure of A intersect E plus the outer measure of E A intersect E complement. This is the definition of being a Lebesgue measurable set.

So in some sense, E is a well-behaved set if it cuts A into reasonable sets. What's the best way to say that? I guess that's OK. So a set is measurable if and only if for all A, we have this equality here.

So let me make a few remarks. First off, this is the left-hand side. No matter what A and E are, the left-hand side is always less than or equal to the right-hand side by the theorem we have up there, that the outer measure of the union is less than or equal to the sum of the measures.

So since for all A E, A is contained in, or in fact, equal to A intersect E union A intersect E complement, we get that the outer measure of A is always less than or equal to the outer measure of A intersect E plus the outer measure of A intersect the complemented of E. This is regardless of A and E. I mean, if E is measurable or not, this always holds.

So since this always holds, we could state being measurable just by, instead of equality, satisfying one of the inequalities. Thus, E is measurable if for all subsets of R, we have that A is less than or equal to the outer measure of A. Because like I said, we know no matter what A and E are, this holds. So if I want equality, then I have to have this as less than or equal to that. So E is measurable if and only if this is less than or equal to the outer measure of A.

So now we have the silliest-- well, let's not state it as an example, but let's we can state it as a theorem. It will be a silly theorem, one which I will not write the actual proof of. But the empty set-- so again, I've told you before that when I'm writing on the board, writing in my notes, I typically shorten words. So "measurable" you will see written as "mble" throughout the notes, and also when I write on the board. I forget the word for what it means to shorten words.

So the empty set is measurable. R is measurable. And we have the fact that a subset of R is measurable if and only if its complement is measurable because either one of these definitions-- remember, by what we've proven about outer measure, this equality is equivalent to requiring this inequality. These are symmetric in E and E complement. So E is measurable if and only if its complement is measurable.

All right, so these are kind of the stupidest measurable sets you could have. Again, the empty set-- why? Because this is then empty and the outer measure of the empty set is 0.

And then over here, I would just get the measure of A intersect the empty set complement, which is R. So I get the outer measure of A is equal to the outer measure of A. And since the empty set is measurable, its complement R is also measurable.

So let's do some non-stupid examples of measurable sets, still kind of trivial because they don't make up very much. But if a set has outer measure 0, then it's measurable. So how do we prove this?

Again, we just need to prove this inequality here for every subset of R. So let A be a subset of R. Then A intersect E, this is contained in E. And therefore, since we know outer measure is-- so the fancy word for when A is a subset of B, the outer measure of A is less than or equal to the outer measure of B-- the fancy word for that is "monotonicity."

But since the outer measure is monotonic, that tells me that the outer measure of E is less than or equal to the A intersect E is less than or equal to the outer measure of E. And since that is 0, that implies that the outer measure of A intersect E is 0. Because the outer measure is always a non-negative number.

Thus, the outer measure of A intersect E plus the outer measure of A intersect E complement-- this is 0, so this is equal to the outer measure of A intersect E complement. A intersect E complement is contained in A. So the outer measure of A intersect E complement is going to be less than or equal to the outer measure of A, again, because this set is contained within A. And thus, E is measurable.

So right now, after two theorems, we've shown that the uninteresting sets are measurable, if by "uninteresting" we mean having very little measure, even though I haven't defined measure yet. I've just defined outer measure. I'm using these terms interchangeably, unfortunately, but you'll see why either by the end of this class or the end of next class.

But so far, we don't have very many interesting examples of measurable sets. What we will show is that there's a lot of interesting measurable sets. In fact, every open set is measurable, and since if an open set is measurable its complement is measurable, and the complements of open sets are closed sets, then we have that every closed set is measurable, as well.

But in fact, it's much richer than that, as we'll see. You can take a countable collection of open sets and take their intersection. That's not necessarily an open set, so it's not clear if it's measurable by what I just said.

But it turns out that this intersection of open sets will be measurable. And again, by taking complements, you get unions of closed sets, which are not necessarily closed, are also measurable. So when I learned measure theory, my instructor told me that if you can write down the set, chances are it's measurable. If you can sit down and write down a union of intersections of complements of so on and so on of some basic sets, then that's measurable. And we'll see why that's true shortly.

Now, before we get to showing everything I just told you, we need some general facts about measurable sets, about the structure of the collection of measurable sets. Right now, we just know that the collection of measurable sets includes the empty set R and the sets that have outer measure equal to 0, which I see I didn't write. This is also the danger of lecturing in an empty classroom is that if I make a mistake on the board, there's no one to correct me.

So next, prove the following theorem about measurable sets-- that if I have two measurable sets, their union is measurable. If E1, E2 are measurable, then their union, E1 union E2, is measurable. So again, we have to verify that inequality to show that it's measurable.

So let A be a subset of R. So since E2 is measurable, we get that the measure of A intersect the complement of E1 is equal to the outer measure of A intersect E1 complement intersect E2 plus the outer measure of E of A intersect E1 complement intersect E2 complement. You may be asking, where the hell did this come from? Why do I care?

Well, it's because I'm getting something. Remember, this is equal to, by De Morgan's law, the complement of the union of E1 and E2. So I want to somehow have some relation involving the complement of E1 union E2 intersect A.

Because again, I'm trying to show that the measure of A intersect E1 union E2 plus the outer measure of A intersect the complement of E1 union E2, which is exactly this term, equals the measure of A. So it looks like this relation was just grabbed out of left field. But that's the thinking behind why you would care.

So now, A, if I take A and intersect it with E1 union E2, this is equal to A intersect E1 union A intersect E2. Now, everything from here that also has something in common with E1 is contained in this set. So this union is actually equal to A intersect E1 union A intersect E2 intersect E1 complement. Now, this appeared here, so you can kind of see maybe some magic is going to start to happen in a minute when we start taking the measure.

So what do we get? We get that the measure of A intersect E1 union E2, which is this side, this is less than or equal to the outer measure of A-- so this is equal to this union, so the outer measure of this is less than or equal to the sum of the outer measure of this and the outer measure of this. Now, we use the fact that E1 is measurable.

So since E1 is measurable, the outer measure of A intersect E1 is equal to the outer measure of A minus the outer measure of A intersect E1 complement. Or I should say that's backwards. And then plus, still, this outer measure here.

But now, we're in good shape because what do we have? We have this term here appearing here. I also have this term here is appearing here.

So subtracting this over here and subtracting that over there, I get that the right-hand side is equal to the outer measure of A minus the outer measure of A intersect. And I'll just rewrite this intersection of complements as the complement of the union by De Morgan's law. And remember, I started off with the outer measure of A intersect E1 union E2, and I showed its less than or equal to the outer measure of A minus the outer measure of A intersect the complement, and therefore, the outer measure of A intersect E1, E2 plus the outer measure of A intersect E1 union E2 complement is less than or equal to the outer measure of A. And that's what we wanted to prove, and therefore, E1 union E2 is measurable.

Now, if you can do something for two things, you can do something for n things, typically, by an induction argument. So the previous theorem implies the following-- that if E1 up to E n are measurable, then this finite union k equals 1 to n E k is measurable. And how do you prove this? You prove it by induction.

So proof by induction-- when n equals 1-- the base case-- this is clear. Suppose-- so call this claim star-- suppose this claim star holds for n equals m. Now we want to show it holds, that this implies that the claim of the theorem holds with n equals m plus 1.

So let E1 E m plus 1 be measurable. Then the union k equals 1 to m plus 1 of E k is equal to union k equals 1 to m of E k union or E sub m plus 1. Now, by the induction hypothesis, a collection of m measurable sets, their union is measurable, so this is measurable by the induction hypothesis.

And we're assuming this is measurable. And then by the previous theorem, the union of two measurable sets is measurable. And that proves the theorem.

So up to this point, we've shown two basic things about measurable sets-- really three. First off, it's nonempty. It's a nonempty collection of measurable sets.

We've shown that a set is measurable if and only if its complement is measurable. And we've also shown that finite unions of measurable sets are again measurable. And therefore, if I look at the collection of all measurable sets, this has a very special structure or a very general type of structure, which I'm now going to elaborate on.

So let's take a pause here about measurable sets because now, we're going to say a few general things about certain classes of sets or certain classes of collections of sets. So let me make a definition here. A nonempty collection of sets-- so this is a collection of subsets of R, so it's a subset of the power set of R-- is an algebra.

So some of you have taken Algebra or are taking Algebra. So there is a notion of what an algebra is in Algebra. Here, we say a collection of sets is an algebra if in some sense it's closed under taking complements and finite unions. If two conditions are satisfied-- if E is in the algebra, then its complement is in the algebra, and two, if I take finitely many elements in the algebra and finitely many subsets from this algebra of subsets, then this union is also in this algebra.

Now, this is for finite unions. You can ask, what about infinite unions? That's not equivalent to finite unions, meaning that's a stronger condition to impose.

So we have a special name for collections of subsets of that type. We say an algebra A is a sigma algebra-- sigma for, if you like, sum, because we really have a summation in mind-- if also the following, stronger condition is satisfied-- if I have a countable collection of subsets in A, so a countable collection of elements of the algebra, then their union is in the algebra.

So an algebra is closed under taking complements and finite unions. A sigma algebra is closed under complements, and also the stronger condition of taking countable unions. So of course, this condition implies this condition-- actually, that's not yet clear. We'll see.

Although it's not in the definition, we'll see in just a second that, in fact, this implies that the empty set has to be a member of the algebra. And therefore, this condition implies this condition if you just take finitely many of these and then take them to be empty after some finite n. So that's the definition of an algebra. That's the definition of a sigma algebra.

Let me give a few examples real quick or a few remarks about this. First off, by De Morgan's laws, which tells you the complement of a union is an intersect of the intersection of the complements, we get that E1 E n in the algebra implies that E1 intersect-- so their intersection, so this is just for an algebra now-- their intersection, which is equal to the complement of the union of their complements. So since each of these is in the algebra, the complement is in the algebra.

And therefore, the finite union is in the algebra. And therefore, the complement of that is in the algebra. So not only is an algebra closed under taking complements and finite unions, it's also closed under taking intersections, finite intersections.

So thus, if I take some element from my nonempty algebra, the empty set, which is equal to E intersect E complement, that's in the algebra. So let me pause for a minute and write this a little carefully. If I take an element of the algebra which is supposed to be nonempty, then the empty set, which is equal to E intersect its complement, that's in the algebra.

That's in the algebra. The intersection of two things in the algebra is in the algebra. So that's also in the algebra.

And also, this implies that R, which is equal to the complement of the empty set, is also in the algebra. So for algebras of sets, they always contain the empty set for nonempty ones, which is the only kind we're ever going to care about. They always contain the empty set in R.

And not only are they closed under taking complements, but also finite unions. And just like we proved that for algebras, finite intersections are also in the algebra, you can also prove that for a sigma algebra, countable intersections are also in the sigma algebra. So let me make a point of that. If A is a sigma algebra, then E n, a countable collection of elements of the sigma algebra, implies that this countable intersection is also in the algebra, again by De Morgan's law.

So why am I making all these general definitions now? So what we're going to show soon, at least in the next lecture, we'll soon show that if I define script M as a set of all measurable subsets of R, so this is a collection of subsets of R, that this is a sigma algebra-- one of the things we'll show. Now how we're doing this now is we had these ideals of what a measure should be, and we're building it up from this way, and in the end we're going to come up with this collection of measurable sets, which will have this special structure of being a sigma algebra.

And then our measure will be defined on this sigma algebra of sets. When you go on to general measure theory, that is the input for you. A measure space will then be a collection, a sigma algebra of subsets of a set, with a measure on it. That's your input.

Here, your building, if you like, one of the first nontrivial measures. That's what we're doing, one of the most important measures, really. So we have that definition.

If you've taken my class before, you know that if there's a definition, then we should see an example or two. I said that the set of measurable sets will be a sigma algebra. That's going to take a little work to get to, but we can come up with a few examples already.

So some simple examples-- we have the stupidest example. Always start off with the stupidest examples. That's the way to go.

So the simplest sigma algebra is given by this collection of subsets consisting only of the empty set in R. The next stupidest one is on the other end of the spectrum, where every subset is in this sigma algebra. So these are sigma algebras.

What's a non-stupid one? Let's say we take A to be the set, the collection of all subsets E, such that either E is countable or E complement is countable. So why is this? I claim this is also a sigma algebra.

Why is this a sigma algebra? First off, it's clear that if E is in A then its complement is in A. Because if E is countable, then the complement of its complement is E, is countable. So E in A implies that E complement is A. In other words, this condition is symmetric in E and E complement.

Why is it closed under taking countable unions? So suppose I take a collection of elements from my collection here, A. I want to show the union is in A.

Then if for all n, E n is countable, then the union over n of E sub n is a countable union of countable sets. And therefore, this is countable, which implies this guy is in the set collection A. So this is the case, that all of these are countable if there exists an integer N0 such that E n complement is countable, E n sub 0.

Then, remember what I have to verify is that the union of E n's is in the set A, which means either it's countable or its complement is countable. So if I look at the complement of the union, this is then equal to, by De Morgan's law, the intersection of the complements. And this is contained in one of these guys.

And therefore, this is a subset of a countable set, which implies that this is countable, and which is one of the conditions of A. And therefore, this is in A. So therefore, this collection of subsets of R such that E is countable or the complement of E is countable is a sigma algebra of sets. It's usually referred to as the co-countable sigma algebra.

And one can define a measure on this collection, this sigma algebra. But we're only interested in Lebesgue measure, which is going to be defined on, as we'll see, the sigma algebra of measurable subsets of R, Lebesgue measurable subsets of R. So let's do one more example.

This will just take a minute and it's not too technical. I know it's the end of the lecture. So maybe-- well, I mean, you're at home watching this. You can pause it at some point, get a snack, change pajamas, whatever.

But anyways, here's one last example of a sigma algebra, which is, in fact, a very important sigma algebra, which is the following. Let capital sigma be the collection of all sigma algebras containing all open sets. So let's unwind this definition for a minute.

So you are in this collection of sigma algebras if you are a sigma algebra and you contain all open subsets of R. So for example, the power set of R, which is a sigma algebra-- a trivial one-- contains every subset of R, so it certainly contains every open subset of R. This is in this collection of sigma algebras.

And let me define script B to be the intersection over all sigma algebras in this collection of sigma algebras. So this is the intersection of every sigma algebra containing all open sets. So first off, this is nonempty. This is a nonempty intersection, so note, this is a intersection of subsets of collections of subsets of R

So this is R and it's nonempty. Then B is-- I'm going to say a lot in these next few words-- is the smallest sigma algebra containing all open subsets of R. And we call it the Borel sigma algebra.

So how do you think of this? You take every sigma algebra in the universe that contain every open set. You take the intersection of all these sigma algebras. My claim is that you get a sigma algebra, and this is the smallest sigma algebra containing all open subsets of R.

So another way to say this last sentence is, E is in sigma and for all A in sigma, E is contained in A. So it's the smallest sigma algebra containing all open subsets of R. The proof is not hard. You just need to make sure you understand exactly what I'm saying here.

And if you understand what I'm saying here, the proof is quite simple. And I'm only going to do one part. So first off, once I show that B is a sigma algebra, then the rest follows because if I take any open subset of R, it's contained in every one of these. And therefore, it's contained in the intersection.

So every open subset is contained in B or every open subset is an element of B. I shouldn't say contained, but every open subset is an element of B. And so I just need to show that B is a sigma algebra.

And since it's equal to the intersection of all sigma algebras containing every open subset, it has to be the smallest one. Any other one-- this intersection for any fixed one, this intersection is contained in any fixed one. And therefore, B is contained in any fixed sigma algebra which contains all the open subsets.

So I just need to verify that B is a sigma algebra. And since I'm running out of time, I'm going to do one part, or verify just one part of the definition. Because the other part's essentially the same except you have to use more chalk. So I just need to verify that B is a sigma algebra.

Now, suppose E is an element of B. So this means it's a subset of R. It's one of these subsets of R that's an element of B.

Then for all sigma algebras in this collection, since B is the intersection over all of these sigma algebras, E is in A. And since each of these is a sigma algebra, I get that the complement is in A. And this statement says for every sigma algebra in this collection, the complement is in that sigma algebra, and therefore, E complement is in the intersection, which is, remember, how I've defined the Borel sigma algebra.

And again, the proof of being closed under countable unions is sort of the same. Take a countable collection of elements of B. Then this countable collection-- they all have to be elements of A for every A in sigma.

Therefore, their union has to be an element of A for every A in sigma because A is a sigma algebra. And therefore, the union is in the intersection over all A's which is equal to B, the Borel sigma algebra.

So what we're working towards and what we're going to do next lecture is we're going to show that M, the collection of Lebesgue measurable subsets of R is a sigma algebra and it contains the Borel sigma algebra. So it contains this very big collection of subsets of R.

I mean, this is truly a very big subset of R, like I was saying. It contains all open subsets. And since it's a sigma algebra, it contains the complements of all open subsets, i.e. closed subsets. but then it also contains all intersections of open subsets because sigma algebras are closed under taking countable intersections, as well.

And then, you could take countable unions of those countable intersections of open subsets, and then, and so on, and so on. So you get a very rich class of subsets of R that's contained in the sigma algebra. And what we're going to show is that, like I said, the collection of Lebesgue measurable subsets of R, that's a sigma algebra and it contains the Borel sigma algebra.

So it's a very rich class of subsets, even though up to this point, all we've shown is that the empty set R and the sets with outer measure equal to 0 are Lebesgue measurable. So we'll do that next time. And I'll stop there.