21. Homogeneous Dynamics II

May 1

In this lecture, we give some more details about how projection theory can help understand homogeneous dynamics. We sketch proofs in a simple case. Then we discuss recent work by Lindenstrauss-Mohammadi which uses projection theory to prove quantitative bounds in Ratner-type equidistribution theorems. The projection theory that appears here is related to some recent problems in projection theory raised by Fassler-Orponen.

We pick up from the end of the last lecture. At the end of the last lecture, we drew a picture to illustrate how $L_{a_r^{-1}}$ acts on the space $X = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. The key point in the picture is that the compressing direction v_{comp} is twisting relative to the unipotent orbits.

To start this lecture, we formulate precisely what we mean when we say that v_{comp} is twisting and indicate how to compute and prove this twisting using matrix computations. Then we explain how to use this twisting to prove bounds about an orbit Ux.

We call $v_{comp}(t) \in T_{u_t g_0}$ the direction that was compressed when we apply $L_{a_r^{-1}}$, i.e. the smallest singular value vector for $dL_{a_r^{-1}}$. We also define an orbit vector $v_{orb(t)}$ such that $u_t(g_0 + \epsilon v_0) = u_t g_0 + \epsilon v_{orb}(t)$. The moral the the story is that at each point there is an orbit vector and a compression vector and the angle between them is changing.

Let us first compute v_{comp} at $g_0 \in G$. Here $v_{comp} = v_{comp,g_0} \in T_{g_0}G$ is the smallest singular vector for $dL_{a_r^{-1}} : T_{g_0}G \to T_{a_r g_0}G$.

To study this we make use of the fact that m is right invariant. So the singular values and vectors of $L_{a_r^{-1}}$ are closely related to those of

$$R_{(a_r g_0)^{-1}} \circ L_{a_r^{-1}} \circ R_{g_0} h = a_r h g_0 (a_r g_0)^{-1} = a_r h a_r^{-1} = C_{a_r} h.$$

Here $C_{a_r} : G \to G$ mean conjugation by a_r. Note that $C_{a_r} : e \mapsto e$. $dC_{a_r} : T_e G \circlearrowright$

$$dC_{a_r} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e^r & \\ & e^{-r} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{-r} & \\ & e^r \end{bmatrix}.$$

Recall: Orthonormal basis for $T_e G$:

$$n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tilde{n} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tilde{d} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

After some calculation, we see that

$$dC_{a_r}(n) = e^{2r} n, dC_{a_r}(\tilde{n}) = e^{-2r} \tilde{n}, dC_{a_r}(d) = d$$

Thus, $R_{g_0}(\tilde{n})$ is the singular vector of $dL_{a_r^{-1}}$ at g_0 with singular value e^{-2r}. In other words

$$v_{comp,g_0} = R_{g_0}(\tilde{n}).$$

Next we want to study $v_{comp}(t) = v_{comp,u_t g_0}$. Plugging in, we get

$$v_{comp}(t) = R_{u_t g_0}(\tilde{n}) = \tilde{n} u_t g_0.$$

For comparison, we consider a vector that tracks the orbits of U. We define an orbit vector $v_{orb(t)}$ such that

$$u_t(g_0 + \epsilon v_0) = u_t g_0 + \epsilon v_{orb}(t).$$

If we use coordinates so that the orbits are vertical lines $\{x\} \times [0,1]$, then in these coordinates $v_{orb}(t)$ will be constant in t. Solving the equation above, we see that

$$v_{orb}(t) = u_t v_0.$$

If $v_{orb}(0) = v_0 = v_{comp}(0) = \tilde{(n)} g_0$, then we would have

$$v_{orb}(t) = u_t \tilde{(n)} g_0.$$

Comparing formulas for $v_{comp}(t)$ and $v_{orb}(t)$ we see that they are not the same. And so the compression direction is twisting relative to the orbits.

Tracking the spread of an orbit

$$U_{[0,T]} x = a_R U_{[0,1]} a_R^{-1} x.$$

Put $\tilde{x} = a_R^{-1} x$ and assume that $\tilde{x}$ is not deep in the cusp. This implies that $U \cdot x$ is not close to being periodic. Let $R = Jr$ and put $U_j = a_r^j U_{[0,1]} \tilde{x}$. Define $|X|_\delta$ to be the nubmer of δ balls needed to cover X. **Goal:** Estimate $|U_j|_\delta$ in terms of j, δ, r. Define X_j to be the top layer of U_j, then $|U_j|_\delta = \delta^{-1} |X_j|_\delta$. We say that we are in the very spread situation if $|U_j|_\delta \sim \delta^{-3}$ and $|X_j|_\delta \sim \delta^{-2}$.

Using the Key Picture

Lemma 21.1. *If $e^{2r} = \delta$. Then,*

$$|X_{j+1}|_\delta \sim \sum_{\substack{t \in \delta\mathbb{Z} \\ 0 \le t \le 1}} \sim \delta^{-1} Avg_{0 \le t \le 1} |\pi_t X_j|_\delta.$$

Let f_t be $L_{a_r^{-1}}$ restricted to time t. f_t looks like the projection map π_t (see Figure 18). f_t is not linear but is smooth.

We now bring into play a rather simple projection estimate.

Proposition 21.2. *If $X \in B_1^2$, then*

$$Avg_{\theta \in S^1} |\pi_\theta X|_\delta \gtrsim |X|_\delta^{1/2}.$$

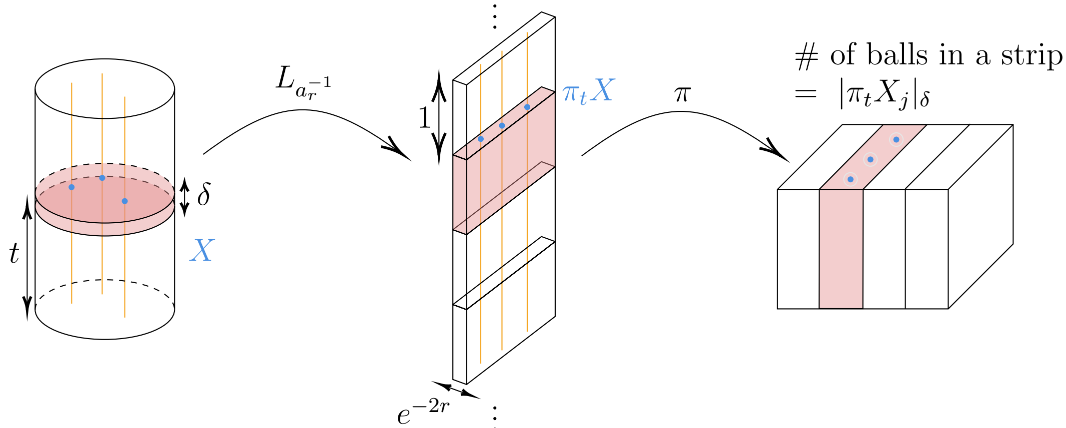

FIGURE 17. Proof sketch for the projection estimate.

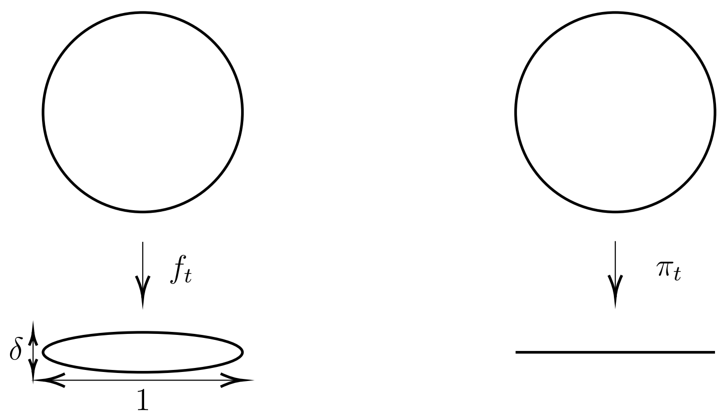

FIGURE 18. f_t is almost a projection.

The sharp case for this example is shown in Figure 19.
The proof for π_t holds for f_t as well.

Corollary 21.3.

$$|X_{j+1}|_\delta \gtrsim \delta^{-1}|X_j|^{1/2}.$$

Proof. $|X_{j+1}|_\delta \gtrsim \delta^{-1} Avg_t |f_t X_j|_\delta \gtrsim \delta^{-1}|X_j|_\delta^{1/2}$. □

Suppose $|X_0|_\delta = 1$, then $|X_1|_\delta \gtrsim \delta^{-1}$, $|X_2|_\delta \gtrsim \delta^{-3/2}$, $|X_3|_\delta \gtrsim \delta^{-7/4} \cdots$

Remark 21.4. *This proof sketch shows that X_j is well spread, but it doesn't show that the orbit is dense.*

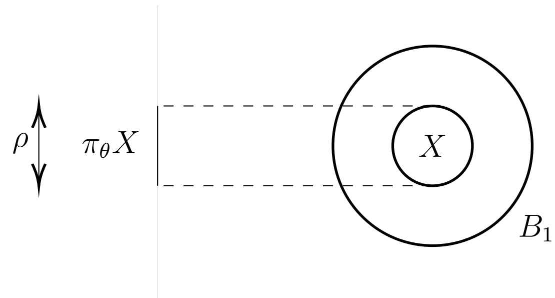

FIGURE 19. Example with $X = B_\rho \subset B_1$, we see that $|X|_\delta \sim (\frac{\rho}{\delta})^2$ and $|\pi_\theta(X)|_\delta \sim \frac{\rho}{\delta}$

This finishes our discussion of homogenenous dynamics ni $SL_2(\mathbb{R})$. Next we consider higher dimensions. Hedlund's theorem was extended to higher dimensions by Dani, Margulis, and Ratner. One key result in the theory is Ratner's theorem. A special case of Ratner's theorem says that if $U \subset SL_n(\mathbb{R})$ is a unipotent subgroup, and $X = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$, then the closure of an orbit Ux is either all of X or is a lower-dimensional homogeneous space.

As one concrete example, we can consider, $G = SL_3(\mathbb{R})$, $\Gamma = SL_3(\mathbb{Z})$. Put,

$$U = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

The orbit closures in this situation were studied by Margulis in connection with the Oppenheim conjecture about the values of quadratic forms.

Ratner's theorem gives the best possible qualitative information about orbit closures in great generality. But there are interesting open questions about quantitative information. We can consider a finite piece of the orbit of the form $U_{[0,T]}x$. In terms of T, it would be interesting to describe how this piece of orbit is distributed in X. Recently, Lindenstrauss, Mohammadi, and collaborators proved strong quantitative bounds about the distribution of $U_{[0,T]}x$ in certain Lie groups. Together with Wang and Yang they gave strong quantitative bounds for the unipotent group $U \subset SL_3(\mathbb{R})$ mentioned above, establishing a strong quantitative version of the Oppenheim conjecture.

In the course of this work, they found a new connection between homogeneous dynamics and projection theory. The discussion above applies their ideas in the much simpler case of $SL_2(\mathbb{R})$.

In the last short section, we explain what is similar and what is different in $SL_n(\mathbb{R})$ for $n \geq 3$.

The initial setup with diagonal matrices and unipotent matrices is quite similar. To study the group $U \subset SL_3(\mathbb{R})$ above, we set

$$a_r = \begin{bmatrix} e^r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-r} \end{bmatrix}$$

Then we have

$$a_r u_t a_r^{-1} = U_{e^r t},$$

which is closely analogous to the setup in $SL_2(\mathbb{R})$.

Next we can study the action of $L_{a_r^{-1}}$. We can study the singular value of $L_{a_r^{-1}}$ by studying the singular value of C_{a_r}. They are

$$e^{-2r}, e^{-r}, e^{-r}, 1, 1, e^r, e^r, e^{2r}.$$

The direction tangent to U is a singular vector with singular value e^r. The perpendicular space is 7-dimensional.

Recall that in the $SL_2(\mathbb{R})$ case, the singular values of $L_{a_r^{-1}}$ were $e^{2r}, 1, e^{-2r}$, and the tangent vector to U is singular vector with singular value e^{2r}. The perpendicular space is 2-dimensional, and the singular values for that space are 1 and e^{-2r}.

The first difference in $SL_3(\mathbb{R})$ is that the perpendicular space is higher dimensional and it has more different singular values. A linear map with singular values 1 and e^{-2r} can be approximated by a projection. In 7 dimensions, a linear map with singular values $1, 1, 1, 1, e^{-r}, e^{-r}, e^{-r}$ can be approximated by a projection from $\mathbb{R}^7$ onto a 4-dimensional subspace. But here, we have to deal with a linear map with singular values $e^{-2r}, e^{-r}, e^{-r}, 1, 1, e^r, e^{2r}$. This linear map is not approximately a projection. However, this is not the most serious issue.

Our key geometric input is that as t varies, the linear map on the perpendicular space twists. In the case of $SL_2(\mathbb{R})$, we get a 1-parameter family of linear maps. Each linear map is almost a projection, and so we almost get the whole set of projections from $\mathbb{R}^2$ to a 1-dimensional space. For $U \subset SL_3(\mathbb{R})$, the variable t still lives in $\mathbb{R}$ because the group U is 1-dimensional, and so we get a 1-parameter family of linear maps on $\mathbb{R}^7$. These linear maps are a bit more complicated than projections, but suppose for a moment that we had a 1-parameter family of projections from $\mathbb{R}^7$ to 4-dimensional subspaces. This 1-parameter family is still a very small subset of all the projections from $\mathbb{R}^7$ to 1-dimensional subspaces. This is the most serious difference between $SL_2(\mathbb{R})$ and $SL_3(\mathbb{R})$.

This leads to a problem called the restricted projection problem, which was posed by Fassler-Orponen. In the restricted projection problem, instead of considering all the projections from $\mathbb{R}^n$ to k-dimensional subspaces, we consider only a smooth lower dimensional family of projections. There are many different choices we could make for this smooth family, leading to many different problems. The simplest interesting example occurs in three dimensions.

Question 21.5. *(Fassler-Orponen 2013)*
For $\theta \in S^2$,, let $\pi_\theta : \mathbb{R}^3 \to \theta^\perp$ be the orthogonal projection. Let γ be curve in S^2. If $X \subseteq B^3$, and X is a (δ, s, C) set, estimate $Avg_{\theta\in\gamma}|\pi_\theta(X)|_\delta$.

The answer depends on whether γ lies in an equator or not.

Example 21.6. *Let γ be the equator and X a $\delta \times 1 \times 1$ slab. Then, $Avg_{\theta\in\gamma}|\pi_\theta(X)|_\theta \sim \delta^{-1}$.*

An equator is a geodesic in S^2 and so it has zero extrinsic curvature in S^2. We say that $\gamma \subset S^2$ is non-degenerate if it has non-zero extrinsic curvature at every point. For non-degenerate curves, there are much stronger estimates.

Theorem 21.7. *(Gan-Guo-Guth-Harris-Maldague-Wang)*
If $X \subseteq B^3$ is a $(\delta, 2, C)$ set and γ is non-degenerate. Then, $Avg_{\theta\in\gamma}|\pi_\theta(X)|_\delta \geq C_\epsilon \delta^{-2+\epsilon}$ for any $\epsilon > 0$.

The proof is based on decoupling in Fourier analysis.

Results about the restricted projection problem in the spirit of the theorem above were used as tools in the work on quantitative Ratner theorems. Here is a sample theorem in this direction:

Theorem 21.8. *(Lindenstrauss, Mohammadi, Wang, Yang, vague statement) There is a constant $c > 0$ so that the following holds. If $G = SL(3, \mathbb{R})$, $\Gamma = SL(3, \mathbb{Z})$. U as above and $U \cdot x$ is not close to a proper homogeneous subspace, then, $U_{[0,T]}x$ is T^{-c}-dense in (G/Γ).*

One key step in the proof of this theorem is that, for $\delta = T^{-c}$,

$$|U_{[0,T]}x|_\delta \geq c_\epsilon \delta^{-(\dim G+\epsilon)}.$$

Part of the proof of this key step follows the ideas we have outlined, but with the restricted projection theorem in place of the simple projection theorem that we used above.

The full proofs of the results we have discussed in homogeneous dynamics require more tools and ideas from homogeneous dynamics. But hopefully these notes give an idea of how tools from projection theory can help to study dynamics.

There is some other recent work in this area by Benard-He and Benard-He-Zhang, applying tools from projection theory to study random walks on homogeneous spaces. The introductions to those papers are a good next step for further reading.