18.156, Projection theory, problem set 1

1. Digesting material from class

Notation. Recall that we write $A \lesssim B$ to mean that there exists a constant c so that $A \le cB$.

1. (Practicing Cauchy-Schwarz and double counting) Suppose that X, Y are finite sets and $f : X \to Y$. If $|Y| \le (1/2)|X|$ prove that

$$\#\{x_1, x_2 \in X : f(x_1) = f(x_2)\} \gtrsim |X|^2 |Y|^{-1}.$$

We recall the convenitions and basic theorems about the Fourier transform over finite fields.

Let $\mathbb{F}_q$ be the finite field with q elements, where $q = p^r$ and p prime.
For $x, \xi \in \mathbb{F}_q^d$, we define $x \cdot \xi = x_1\xi_1 + ... + x_d\xi_d$.
We let $e : \mathbb{F}_q \to \mathbb{C}^*$ be a non-trivial group homomorphism, from $\mathbb{F}_q$ with addition to $\mathbb{C}^*$ with multiplication.
(If $q = p$ is prime, then we can define $e(x) = e^{2\pi i \frac{x}{p}}$. But any choice of e works equally well. If $q = p^r$, the image of e will be the p^{th} roots of unity.)
Suppose $f : \mathbb{F}_q^d \to \mathbb{C}$. For any $\xi \in \mathbb{F}_q^d$, define

$$\hat{f}(\xi) = \sum_{x \in \mathbb{F}_q^d} f(x) e(-x \cdot \xi).$$

Theorem 1. *(Fourier inversion) If $f : \mathbb{F}_q^d \to \mathbb{C}$, then*

$$f(x) = \frac{1}{q^d} \sum_{\xi \in \mathbb{F}_q^d} \hat{f}(\xi) e(x \cdot \xi).$$

Theorem 2. *(Plancherel) If $f, g : \mathbb{F}_q^d \to \mathbb{C}$, then*

$$\sum_{x \in \mathbb{F}_q^d} f(x)\overline{g(x)} = \frac{1}{q^d} \sum_{\xi \in \mathbb{F}_q^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}.$$

2. On your own, work through the proof of Theorem 1 and Theorem 2. (You don't have to turn anything in.) The proofs are based on linear algebra and on orthogonality. Here is a sketchy outline.

a. Because $e : \mathbb{F}_q \to \mathbb{C}^*$ is a non-trivial group homomorphism, $\sum_{x \in \mathbb{F}_q} e(x) = 0$.

b. Building on part a, show that for any non-zero ξ in $\mathbb{F}_q^d$, $\sum_{x \in \mathbb{F}_q^d} e(x \cdot \xi) = 0$.

c. Building on part b, show that for any $\xi_1, \xi_2 \in \mathbb{F}_q^d$,

$$\frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} e(x \cdot \xi_1) \overline{e(x \cdot \xi_2)} = \begin{cases} 1 & \xi_1 = \xi_2 \\ 0 & \text{else} \end{cases}$$

d. Show that the set of functions $\{\frac{1}{q^{d/2}} e(x \cdot \xi)\}_{\xi \in \mathbb{F}_q^d}$ forms an orthonormal basis of $\ell^2(\mathbb{F}_q^d)$.

e. Theorems 1 and 2 follow from d with a bit of algebra.

3. Suppose that $P \subset \mathbb{F}_q^d$ is an affine k-plane. Define the perpendicular subspace $P^\perp$ by

$$P^\perp = \{\xi \in \mathbb{F}_q^d : (x_1 - x_2) \cdot \xi = 0 \forall x_1, x_2 \in P\}.$$

Show that

$$|\hat{1}_P(\xi)| = \begin{cases} q^k & \xi \in P^\perp \\ 0 & \text{else} \end{cases}$$

In class we proved some projection estimates for sets $X \subset \mathbb{F}_q^2$ using Fourier analysis. In this problem, you will generalize these estimates to $X \subset \mathbb{F}_q^3$

Let $Gr_q(k, d)$ be the set of k-dimensional subspaces $W \subset \mathbb{F}_q^d$.

For each $W \in Gr_q(d-k, d)$, let $\pi^W : \mathbb{F}_q^d \to \mathbb{F}_q^k$ be a linear map with kernel W.

4. Suppose that $X \subset \mathbb{F}_q^3$. Suppose that $D \subset Gr_q(1, 3)$. Let $S = S(X, D) = \max_{W \in D} |\pi^W(X)|$. If $S \le q^2/2$, prove that

$$|D| \lesssim \frac{S q^2}{|X|}. \tag{1}$$

Then on your own, check that this gives sharp bounds in the following two cases.

Case 1. If $D = Gr_q(1, 3)$, then we get $|X| \lesssim S$, which is sharp.

Case 2. If X is a 2-plane in $\mathbb{F}_q^3$ and D is the set of directions tangent to the 2-plane.

2. Optional exploring futher

Here are some options for exploring further if you would like to do so.

In each problem set, I will include some options to explore. At the end of the class, everyone will do a final project. One option for a final project is to study one of these further exploration questions and write up what you learn about it.

Suppose that $X \subset \mathbb{F}_p^3$ and D and S are as above. Given $|X|$ and S, how big could $|D|$ be? Try out a few examples and see if you can make a plausible conjecture. If you want, you could also consider the same problem over $\mathbb{F}_q$, where there are more examples based on subfields.

The example where X is a 2-plane plays an important role in the last story. You could consider sets $X \subset \mathbb{F}_q^3$ where $|X \cap P| \leq B$ for any 2-plane $P \subset \mathbb{F}_q^3$. Given $|X|$ and S and B, how big could $|D|$ be? Again you could try some examples. Can you prove a bound that improves on problem 4 when B is much less than q^2?

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