18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Last lecture, we found that we could write the reduced Laplacian matrix from the matrix tree theorem as

$$\tilde{L} = \tilde{B} \cdot \tilde{B}^T,$$

where $\tilde{B}$ is our oriented incidence matrix with one row removed. This means that $\tilde{B}$ is an $(n-1) \times m$ matrix, where n is the number of vertices in our graph and m is the number of edges.

By the Cauchy-Binet formula, this can be written as a sum. If E is the set of edges,

$$\det(\tilde{L}) = \det(\tilde{B} \cdot \tilde{B}^T) = \sum_{\substack{S \subset E \\ |S| = n-1}} \left(\det \tilde{B}^S\right)^2$$

(where the square comes from $\det A = \det A^T$ for a square matrix). Our goal is to show that this is the number of spanning trees!

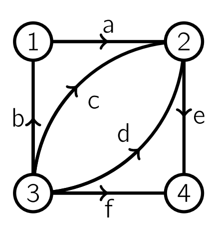
Lemma 1

The determinant

$$\det \tilde{B}^S = \begin{cases} \pm 1 & S \text{ is the edges of a spanning tree} \\ 0 & \text{otherwise.} \end{cases}$$

Note that this lemma finishes the problem, because each spanning tree gives $(\pm 1)^2 = 1$, and everything else gives 0.

Proof. For example, consider our graph G:



This has the oriented incidence matrix (with columns a, b, c, d, e, f in that order)

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

and let's let $\tilde{B}$ be B with the last row removed. Now note that the determinant

$$\det(\tilde{B}^{\{a,b,c\}}) = 0$$

because the graph formed by a, b, c is not a tree. Specifically, this is because we have a cycle! The column vectors $\vec{a} + \vec{b} - \vec{c} = 0$, and therefore the columns are linearly dependent.

In general, if we take a cycle with $C = e_1, \cdots, e_k$, pick some orientation of the cycle. Then there will be some relation

$$\pm \vec{e_1} \pm \vec{e_2} \cdots \pm \vec{e_k} = 0,$$

where the $\pm$s are $+$ if the edges agree with C's orientation and $-$ otherwise! So the rows are dependent, and now we've proved the "otherwise" part of the claim.

Now let's try to investigate the others: if we take the determinant

$$\det(\tilde{B}^{\{a,c,f\}}) = \det \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \pm 1.$$

One way to think of this is that we can always pick a vertex which is a leaf, so that will only have one entry! We can do row and column operations to make that the top left corner, and now we can just use induction.

Formally, let S be some edges that form a spanning tree. All trees on at least two vertices have at least two leaves, so pick any leaf ℓ such that ℓ is not the index of the removed row. Then the ℓth row of the matrix $\tilde{B}^S$ has only one nonzero entry, and it is a ± 1. Expanding out the determinant by this row, the determinant

$$\det(\tilde{B}^S) = \pm \det B',$$

where B' is an $(n-2) \times (n-2)$ matrix. This corresponds to a smaller tree, so we can use induction! Eventually, we get to a tree on 2 vertices, which has determinant ± 1, and we're done! □

Let's think about some generalizations: first of all, let's make a weighted version of the matrix tree theorem. Suppose that to every edge e of G, we assign a weight x_e, and define the weight of a spanning tree

$$\text{wt}(T) = \prod_{e \text{ edge of } T} x_e.$$

Note that this is a product over edges, not a product over vertices (in contrast to the Cayley formula)! Analogously to Cayley, though, we can define a polynomial

$$F_G = \sum_{T \text{ spanning tree}} \text{wt}(T)$$

Our goal is to find this polynomial F_G. Similarly, we can also define a weighted Laplacian matrix:

Definition 2

Given a weighted graph $\{G, \{x_e\}\}$, we define the **weighted Laplacian** L_{ij} via

$$L_{ij} = \begin{cases} \sum_{\substack{e \text{ incident to} \\ \text{vertex } i}} x_e & i = j \\ -x_e & e = (i,j), i \neq j \\ 0 & \text{otherwise} \end{cases}.$$

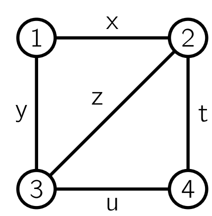
Theorem 3 (Weighted matrix tree theorem)

The polynomial

$$F_G = \det(\tilde{L}),$$

where we remove one row and one column to L.

Let's do an example! If we have a weighted graph



Then our Laplacian is

$$L = \begin{pmatrix} x+y & -x & -y & 0 \\ -x & x+z+t & -z & -t \\ -y & -z & y+z+u & -u \\ 0 & -t & -u & u+t \end{pmatrix},$$

and we claim that any 3×3 minor is actually going to give us the sum of weights of all spanning trees.

Proof. We can think of this graph as a network - specifically, we can think of our weights as conductances of edges. But there's another way to think of this: if the weights are all nonnegative integers, a weight of k could be thought of as k edges between the relevant vertices!

In that case, F_G is just the number of spanning trees of G^m, the graph where each edge e is replaced by x_e unweighted copies of the edge. Thus we can use the regular matrix tree theorem to prove this for all positive integers!

And now the weighted matrix tree theorem holds in general: we have two polynomials that agree on infinitely many points, so they must be the same polynomial. □

Here's another generalization to directed graphs! Let G be a digraph (directed graph) with no loops, but where multiple edges are allowed. There's two different versions of "spanning trees" for our diagraph:

- An **out-tree** oriented at a vertex r is a spanning tree of G in the usual undirected sense, except that for every vertex i, all edges in the shortest path from r to i are directed away from the root.
- An **in-tree** is the same, except that all edges in the path from r to i are directed toward the root instead.

Note that these numbers may not be equal, especially for a given vertex r. But we claim that we can still find them as the determinant of some matrix! We again have two versions of our Laplacian matrix: we define L^{in} and L^{out}:

$$(L^{\text{in}})_{ij} = \begin{cases} \text{indegree}(i) & i = j \\ -\text{number of directed edges } i \to j & i \neq j. \end{cases}$$

$$(L^{\text{out}})_{ij} = \begin{cases} \text{outdegree}(i) & i = j \\ -\text{number of directed edges } i \to j & i \neq j. \end{cases}$$

So here $L^{\text{out}} = D^{\text{out}} - A$, $L^{\text{in}} = D^{\text{in}} - A$, where A is the directed adjacency matrix. (Note that the **outdegree** is the number of edges leaving a vertex, and the **indegree** is the number of edges entering a vertex.)

These vertices may not be symmetric anymore, but they still have some important properties! For example, all column sums of L^{out} are zero, but not necessarily row sums, and vice versa for L^{in}.

Definition 4

Given a square matrix L, define $L^{ij} = (-1)^{ij}\det(L^*)$, where L^* is the matrix L with the ith row and jth column removed.

Theorem 5 (Directed Matrix Tree Theorem)

The number of out-trees rooted at r, is the cofactor

$$(L^{\text{in}})^{ir}$$

for any $i = 1, 2, \cdots, n$, and the number of in-trees rooted at r is

$$(L^{\text{out}})^{ri}$$

for any $i = 1, 2, \cdots, n$.

We'll see how to prove this next time! This is even more general, but the proof is actually easier.