Problem Set 1 Solutions I

Problem 1

Prove a permutation is queue-sortable if and only if it is 321-avoiding.

Solution by Fadi Atieh. First, we prove that if a permutation $w_1, \cdots, w_n$ has a 321-pattern, it is not queue-sortable. In this case, there exist $w_i > w_j > w_k$ where $i < j < k$. We can't put w_i in the list yet, because w_j is smaller than it, so w_i is pushed in the queue. However, w_j also cannot be pushed immediately, so it must go in the queue as well. So w_i will exit the queue first, contradiction.

Now, let's say w is not queue-sortable. Note that for any w, there's a unique way to queue-sort: we have a sorting pointer, which tells us which step we're at, we have a queue, and a partially filled list. Whenever we get to an element a, if there is something smaller than it that still hasn't been put in the list, we must put a in the queue; otherwise, it goes in the list.

Well, if we get stuck, we must be trying to put some element w_i in the list which currently ends in w_k, but $w_i > w_j > w_k$ for some w_j is in the queue. This means $w_i > w_j > w_k$, but w_k came before w_j, which came before w_i, and we've found a 321-pattern. □

Problem 2

Prove the identity

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2.$$

Solution by Agustin Garcia. The left hand side is the generating function for Young diagrams inside $n \times n$ rectangles. Take any such diagram: there is exactly one $k \times k$ box that fits in the upper left hand corner (this is called the **Durfee square**. Now we can split our Young diagram into a $k \times k$ square and then two Young diagrams inside $k \times (n-k)$ rectangles, which are generated by the function $\begin{bmatrix} n \\ k \end{bmatrix}_q$. Tack on a q^{k^2} for the $k \times k$ square, and we're done! □

Problem 3

Show that the number of set-partitions of $[n]$ such that i and $i+1$ are not in the same set for all $1 \leq i \leq n-1$ is the number of set-partitions of $[n-1]$.

Solution by Christina Meng. Looking at the left hand side, we can think of this problem in terms of rook placements: we want to place rooks in a board with rows $n-1, n-2, \cdots, 1$. But if we can't have i and $i+1$ in the same partition, then we can't have any rooks in the bottom corner, making this equivalent to just a rook placement for a board with rows $n-2, n-3, \cdots, 1$, and we're done because this is just the right hand side! □

Solution by Sophia Xia. Construct a bijection between the two sets. Given a partition π of $[n-1]$, we want to map this to a partition of $[n]$ with no two consecutive integers in the same block.

Look at each block in the partition. For every maximal sequence $i, i+1, \cdots, j$ of consecutive integers in a block of π, remove $j-1, j-3, \cdots$, until either i or $i+1$, and place them in a block with n. We can check that this gives a partition of $[n]$ with no two consecutive integers in the same block. To go backwards, look at all the things in the same block as n and put those elements back! Put k in the block with $k+1$. □

Problem 4

Show the identity

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k},$$

where $yx = qxy$, $qx = xq$, $qy = yq$.

Solution by Ganatra. We know that the commutator $[x, y] = xy - yx$ is not zero, but $[q, x] = [q, y] = 0$.

Let $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$, and define $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $n \in \mathbb{N}$ and $k > n$ or $k < 0$. So the valid range is when $0 \leq k \leq n$.

We also have the fact from class

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Proceed by induction. This holds for $n = 0$, and let's assume this holds for all integers $n \leq m$. Then

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$$

and we want to show this is equal to

$$\sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q x^k y^{n+1-k} = \sum_{k=0}^{n+1} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) x^k y^{n+1-k}$$

and this is equal to

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n+1-k} + \sum_{k=1}^{n+1} q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q x^k y^{n+1-k}$$

and shifting the index of summation, this is

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n+1-k} + \sum_{k=0}^{n} q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$$

Switching the terms,

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{n-k} x^k x y^{n-k} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} y$$

but because $q^{n-k}xy^{n-k} = y^{n-k}x$ (by moving the x past the ys one at a time), this is just

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} x + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} y = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} (x+y)$$

as desired. $\square$