18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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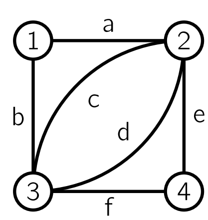
Today, we're going to start talking about the **matrix tree theorem**. Recall, that if we have a graph $G = (V, E)$ with no loops but potentially multiple edges, we define a **spanning tree** to be a subgraph $T = (V, E')$ (taking some subset $E' \subset E$) which is a tree on all $|V|$ vertices. Let $T(G)$ be the number of possible spanning trees: for example, there are 12 of this graph:



There are several matrices we can associate with a graph: let's talk about some of them! For simplicity, let's say we have $|E| = m$ edges, and let's say the vertices are numbered from 1 to n.

Definition 1

The **incidence matrix** of a graph G is

$$C = (c_{ie}),$$

which is an $n \times m$ matrix. $c_{ie} = 1$ if i is incident to e and 0 otherwise.

For example, the incidence matrix of the graph above can be (letting the columns correspond to edges a, b, c, d, e, f)

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Definition 2

The **adjacency matrix** of a graph G is

$$A = (a_{ij}),$$

an $n \times n$ matrix, where a_{ij} is the number of edges connecting i to j.

Note that the diagonal entries are always 0, because we assume we have no loops. In the graph above,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The next definition is the one we really care about:

Definition 3

The **Laplacian matrix** of a graph G is an $n \times n$ matrix

$$L = D - A,$$

where D is a diagonal matrix with $d_i = \deg_G(i)$.

For example, in our matrix above,

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & -2 & 4 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

Notice that by definition, we have undirected graphs, so both the Laplacian and adjacency matrices are symmetric.

Fact 4

All row sums of L are 0, and this means that the determinant of L is 0.

This is because each edge counts once for the degree of any vertex, but also once negatively in the negative adjacency matrix!

Now we're ready to formulate the main result.

Theorem 5 (Matrix Tree Theorem)

Fix $i \in [n]$, and define

$$\tilde{L} = L^i = L \text{ with the } i\text{th row and column removed.}$$

This is known as the **reduced Laplacian matrix**. Then the number of spanning trees of G is just the determinant $\det \tilde{L}$.

In particular, all of those determinants are the same! This is actually a general fact about matrices:

Fact 6

For any symmetric matrix $n \times n$ matrix B with row sums 0, the determinants of B^i and B^j are always equal.

These are called **principal cofactors** of our matrix B. Moreover, if our eigenvalues are $\mu_1, \cdots, \mu_{n-1}, \mu_n = 0$ (we have one zero because of the zero row sum), then this value is actually just

$$\det(B^i) = \frac{\mu_1 \cdots \mu_{n-1}}{n}.$$

This theorem is also called Kirchhoff's theorem, and it is related to Kirchhoff's laws of electricity (found in 1847). But the matrix tree theorem (MTT) was first proved by Borcherdt in 1860, and it's closely related to the work of Sylvester in 1857. Let's do some examples!

Example 7

Let's take the graph G above.

Then the second cofactor (removing the second row and column) of the Laplacian matrix is

$$\tilde{L} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

which has determinant $16 - 2 - 2 = 12$, as we expect! Basically, in general it's best to remove a row and column with lots of edges coming out of it: that's why we prefer to remove vertex number 2 in the example above.

Fact 8

The eigenvalues of our adjacency matrix $A(G)$ are called the **spectrum** of the graph G. **Spectral graph theory** studies properties in terms of this spectrum! But don't confuse the eigenvalues λ_i of the adjacency matrix with the eigenvalues μ_i of the Laplacian matrix. There's only one specific case where we can say something important: if G is d-regular (so all degrees are the same), then $L = dI - A$, so the eigenvalues are all pretty simple: $\mu_i = d - \lambda_i$.

By the way, we can get Cayley's formula from the matrix tree theorem! Remember that if $G = K_n$, the complete graph on n vertices, the Laplacian matrix has all degrees $n-1$ and adjacency matrix all non-diagonal entries 1, so

$$L = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}.$$

Now we just want the determinant of $\tilde{L}$, which just has $n-1$ on the diagonal entries and -1 everywhere else.

Fact 9

Notice that $\tilde{L} - nI_{n-1}$ has all entries -1, so it has rank 1. Thus all eigenvalues of the matrix but one are equal to 0, and the trace of the matrix is $1-n$, so the last eigenvalue is $1-n$.

So if $\tilde{L} - nI_{n-1}$ has eigenvalues $0, 0, \cdots, 0, -n+1$, where there are $n-2$ zeros, then $\tilde{L}$ has eigenvalues $n, n, \cdots, n, 1$, and thus the determinant is just the product of the eigenvalues, which is n^{n-2}, as desired!

In general, the matrix tree theorem doesn't tell us that there will be a nice formula, but we actually get a nice closed form pretty often: here's a related idea to that. Let's say G is a graph on n vertices without multiple edges, and G^+ is obtained from G by adding an extra vertex 0 (thinking of this as a root). Assume that 0 is connected to all other vertices: note that it's possible, for example, that G^+ is connected but G isn't.

So now define a monomial for our graph G:

$$F_G(x) = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G^+}} x^{\deg_T(0)-1}.$$

The number of spanning trees of G is exactly $\frac{F_G(0)}{n}$, because plugging in $x = 0$ only keeps the trees T where 0 is a root, and then there are n ways to attach 0: this is very similar to the Cayley formula proof! Now we can use the following useful fact:

Theorem 10 (Reciprocity Theorem (Bedrosian, 1964))
Let $\overline{G}$ be the complementary graph to G: we take $(V, \overline{E})$, including an edge between two vertices in $\overline{G}$ if and only if it is not included in G. Then

$$F_G(x) = (-1)^{n-1} F_{\overline{G}}(-x-n).$$

This will be left as an exercise! One way is to express this polynomial in terms of the matrix tree theorem: we then just need to show that two determinants are equal. There's also a combinatorial proof. Well, let's try to use this theorem! First of all, we'll prove Cayley's formula again.

Proof. Let O_n be the empty graph on n vertices. It has no spanning trees (or even edges), but O_n^+ now has an extra vertex 0 connected to each of $1, 2, 3, \cdots, n$: this has exactly one spanning tree, so

$$F_{O_n}(x) = x^{n-1}.$$

By the reciprocity theorem,

$$F_{K_n}(x) = (-1)^{n-1}(-x-n)^{n-1} = (x+n)^{n-1}$$

and now plug in $x = 0$: our answer is

$$\frac{n^{n-1}}{n} = n^{n-2}.$$

□

But we can do more! Let's consider the disjoint union of two graphs $K_m \cup K_n$: this means we have m nodes in a clique, all disconnected from another n nodes in a clique. But adding a new vertex 0 connects the two parts: how many trees do we have here? We need to pick a tree of K_m and a tree of K_n, which will be connected our vertex 0. Then 0 gets the factor of -1 twice, so we just correct for it:

$$F_{K_m \cup K_n}(x) = F_{K_m}(x) \cdot F_{K_n}(x) \cdot x^{2-1} = x(x+m)^{m-1}(x+n)^{n-1}.$$

The complement of this graph is the **complete bipartite graph** $K_{m,n}$, and we have a graph on $m+n$ vertices where there are m vertices on the left and n vertices on the right: there are only mn edges between the vertices on the left and right and nothing else.

Now by the reciprocity theorem, we have to replace x with $-x-m-n$, so if we do math

$$F_{K_{m,n}}(x) = (x+n)^{m-1}(x+m)^{n-1}(x+m+n).$$

Calculating the constant term and dividing by the number of vertices, we get the following:

Corollary 11
The number of spanning trees of the complete bipartite graph $K_{m,n}$ is

$$m^{n-1} n^{m-1}.$$

As an exercise, try to prove this combinatorially!