18.212: Algebraic Combinatorics

Andrew Lin

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This class is being taught by **Professor Postnikov**.

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Let's start by finishing the proof of the Hook Length formula: recall that we had variables $x_1, \cdots, x_k, \cdots, y_1, y_\ell$, and we said that given a $k+1$ by $\ell+1$ grid or graph G such that every vertex has a weight $w(v)$ with the following rules:

- The bottom row has weights $\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_k}, 1$.
- The right column has weights $\frac{1}{y_1}, \frac{1}{y_2}, \cdots, \frac{1}{y_\ell}, 1$.
- Any other grid square in the ith column and jth row has weight $\frac{1}{x_i+y_j}$, where $i \le k$ and $j \le \ell$.

We wanted to consider all lattice paths from A (top left corner) to B (bottom right corner), where each path P has a weight

$$w(P) = \prod_{v \in P} w(v).$$

Lemma 1

The sum of all lattice paths

$$\sum_{P:A\to B} w(P) = \frac{1}{x_1 x_2 \cdots x_k y_1 y_2 \cdots y_\ell}.$$

Proof. We induct on $k+\ell$. Base case is $k+\ell = 2$: clearly the vertices have weights $\frac{1}{x_1+y_1}, \frac{1}{y_1}, \frac{1}{x_1}, 1$, and

$$\frac{1}{x_1+y_1} \cdot \frac{1}{x_1} + \frac{1}{x_1+y_1} \cdot \frac{1}{y_1} = \frac{1}{x_1 y_1}.$$

Now any path either goes down from A (to A') or right from A (to A''). Thus the sum over all paths of $w(P)$ is

$$\frac{1}{x_1+y_1} \quad \sum_{P':A'\to B} w(P') + \sum_{P'':A''\to B} w(P'')\Bigg)$$

and by the inductive hypothesis, this is just

$$\frac{1}{x_1+y_1}\left(\frac{1}{x_1 x_2 \cdots x_k y_2 \cdots y_\ell} + \frac{1}{x_2 \cdots x_k y_1 y_2 \cdots y_\ell}\right)$$

which is clearly just

$$\frac{1}{x_2 \cdots x_k y_2 \cdots y_\ell} \cdot \frac{1}{x_1+y_1}\left(\frac{1}{x_1} + \frac{1}{y_1}\right) = \frac{1}{x_1 \cdots x_k y_1 \cdots y_\ell}.$$

□

So remember that in our hook-walks, we're allowed to jump over rows and columns, so this is not exactly what we need in our proof. Recall the exact definition: we start at any vertex in our grid graph, and each time we jump to a vertex that is either to the right of our below our current spot, but we always end at B.

Lemma 2

When we have hook-walks P,

$$\sum_{P \text{ hook-walk}} w(P) = \left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{x_2}\right)\cdots\left(1+\frac{1}{x_k}\right)\left(1+\frac{1}{y_1}\right)\cdots\left(1+\frac{1}{y_\ell}\right).$$

Proof. Expand the product! Each term corresponds to a subgrid: we skip certain rows and columns if we use a 1 instead of a $\frac{1}{x_i}$ or $\frac{1}{y_j}$. Recall that we do not need to start at the top left corner, so we do not need to necessarily include $\frac{1}{x_1}$ and $\frac{1}{y_1}$. Also, the factor of $1 \cdot 1 \cdot \dots \cdot 1$ just starts at B and finishes immediately. $\square$

Now, remember that we defined these hook walks on standard Young tableaux. We start at any box, and we keep traveling to the right and down (skipping squares if we want) until we hit a corner box: a square from which we cannot move down or right.

Remember that we denote $P(u, v)$ to be the probability that we end up at corner box v if we start at u. Any path that ends up at v stays in the rectangular grid between the top left corner and v. This can be written as

$$= \sum_{p:u\to v} \frac{1}{h(u)-1}\frac{1}{h(u')-1}\frac{1}{h(u'')-1}\cdots.$$

Remember that these weights work out the same way as our grid! So we can apply lemma 2. Fix a corner box v: call the set of boxes above or to the left of v the "cohook at v." Then

$$\sum_u P(u, v) = \prod_{t \text{ box in cohook } v} \left(1+\frac{1}{h(t)-1}\right) = \prod_t \frac{h(t)}{h(t)-1}$$

But we claim this is easily simplified. If we remove v from the hook-length, what happens to the product of hook lengths $H(\lambda)$? Nothing, except all hook lengths in the co-hook decrease by one! So this is just equal to

$$\sum_u P(u, v) = \frac{H(\lambda)}{H(\lambda - v)}.$$

So it's time for the punchline: for any fixed box u of λ,

$$\sum_{v \text{ corner}} P(u, v) = 1,$$

since any hook walk ends at some corner. So now adding over all $u \in \lambda$, v corner boxes,

$$\sum_{u\in\lambda}\sum_{v \text{ corner}} P(u, v) = n,$$

since there are n boxes. But now switch the order of summation:

$$\sum_{v \text{ corner}}\sum_{u\in\lambda} P(u, v) = n,$$

and therefore by a calculation above,

$$\sum_{v \text{ corner}} \frac{H(\lambda)}{H(\lambda - v)} = n.$$

But now we can multiply both sides by $(n-1)!$ and rearrange:

$$\frac{n!}{H(\lambda)} = \sum_{v \text{ corner}} \frac{(n-1)!}{H(\lambda - v)}$$

which is exactly the recurrence relation we were trying to prove, and we're done!

Apparently we can think of everything we're doing as a "linear extension of a poset," so now we're changing topics!

Definition 3

A **poset** (partially ordered set) $\mathcal{P}$ is a set of objects with a **binary operation** $\leq$, satisfying the following axioms:

- $a \leq a$ for all $a \in \mathcal{P}$.
- If $a \leq b, b \leq a$, then $a = b$.
- If $a \leq b, b \leq c$, then $a \leq c$.

Note that it is not necessary for either $a \leq b$ or $b \leq a$ to be true. We also define $a < b$ if $a \leq b$ and $a \neq b$.

Definition 4

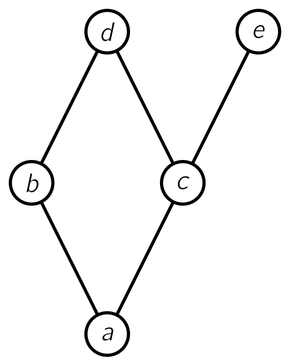
a is **covered** by b if $a < b$ and there does not exist c such that $a < c < b$. We denote this as $a \lessdot b$.

If we're talking about finite posets, this is all we need: all we need is which elements cover each ones. But infinite posets don't quite work: there are no covering relations for the real numbers, for example.

Definition 5

The **Hasse diagram** is the graph formed by covering relations: larger vertices are on the top and smaller vertices are on the bottom.

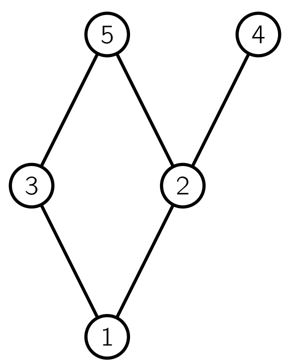
For example, in the diagram here, $a \lessdot b, a \lessdot c, b \lessdot d, c \lessdot d, c \lessdot e$.



Definition 6

A **linear extension** of a finite poset $\mathcal{P}$ with n elements is a bijective map $f : \mathcal{P} \to \{1, 2, \cdots, n\}$ such that $a \leq b \implies f(a) \leq f(b)$.

For example, here's a linear extension for the poset above:



Definition 7

Define $\text{ext}(\mathcal{P})$ to be the number of linear extensions of $\mathcal{P}$.

Turns out we can convert Young tableaux into linear extensions! We can replace boxes with dots and move the top left corner to the bottom. So rotate by 135° counterclockwise, and this means SYTs become linear extensions.

But we have an expression for the number of SYTs: does one exist for posets? The answer is no, but here is an example of a hook-length-type formula!

Theorem 8 (Baby hook length formula for rooted trees)

Given a rooted tree T with n nodes, we can correspond it to a poset $\mathcal{P}_T$. Then

$$\text{ext}(\mathcal{P}_T) = \frac{n!}{\prod_{a \in T} h(a)},$$

where $h(a)$ is the number of nodes including and above a vertex a.

This is an exercise, but it's easier to prove than the other hook length formula!