

[SQUEAKING] [RUSTLING] [CLICKING]

YUFEI ZHAO: The probabilistic method in combinatorics is a powerful method that can be used to demonstrate the existence of some object by introducing randomness. In this video, we'll look at a basic example illustrating an application of the probabilistic method. We'll show this theorem over here, which says that if G is a graph with m edges, then G has a bipartite subgraph with at least $m/2$ edges.

So let me remind you the definition of a bipartite graph is one where I can partition the vertices into two halves or two sets of vertices. And all the edges go from one side to the other. So that's an example of a bipartite graph.

And what this theorem is saying is that if we start with any graph G with some number of edges-- in this case, five edges-- I can find a pretty large subgraph by keeping at least half of the edges of G . For example, we can do this, keeping four of the 5 edges to get this subgraph, G prime.

And this graph, G prime, is now bipartite because I can put these two vertices on one side and these two vertices on the other side, so that you see all the edges go from the red or the pink vertices to the green vertices, and none between the pink and none between the greens. So this is an example of finding a large bipartite subgraph in a graph G .

Now, let me prove this theorem, illustrating the probabilistic method. So we're given this graph G . And what we'll do is assign a color, which will be either black or white uniformly and independently at random to each vertex of the graph G . So imagine flipping a coin for each vertex and coloring each vertex, Black or white. After doing this, we can put all the white vertices of G on one side and all the black vertices of G on the other side.

OK. So now, G has some number of vertices. And there will be some edges going across from white to black, some edges within white, and some edges within black. There could be a lot more edges. But that's an example of what this graph might look like.

And now, let me take G prime to be the subgraph consisting of edges with white on one end and black on the other end, so two different colors on the two edges, two endpoints of this edge. This is a subgraph. It's a subset of the edges of G . And note that every edge of G has probability $1/2$ of being in G prime because, for this edge, there are four different possibilities for what the point colors might look like. And these four possibilities are equally likely. And exactly two of them will create an edge that falls in G prime.

Now, knowing this by linearity of expectations, the number of edges of G prime-- here, G prime is a random subgraph. Where G is given, G prime depends on the coin flips that we made to assign colors. So it's a random subgraph. So the number of edges of G prime is a random variable. But this random variable has an expectation. And we can compute this expectation using linearity of expectations because each edge falls in G prime with probability $1/2$. And therefore, the total expected number of edges in G prime is $m/2$.

This is the expectation. This is what happens on average. Thus, some instance of G prime has at least $m/2$ edges. If on average we get $m/2$ edges, then there must be some instance with at least $m/2$ edges. And this G prime is what we're looking for. It has at least $m/2$ edges. And it is bipartite because we've kept all the edges from the black side to the white side. And so the prime is bipartite.

This finishes the proof of the theorem. And you see that by introducing randomness, we can prove something about the existence of a structure with desired properties. And it's a hallmark of the probabilistic method. And it's an important method that has lots of applications. This is one of the simplest examples of its applications.